Ali Hasan Nayfeh

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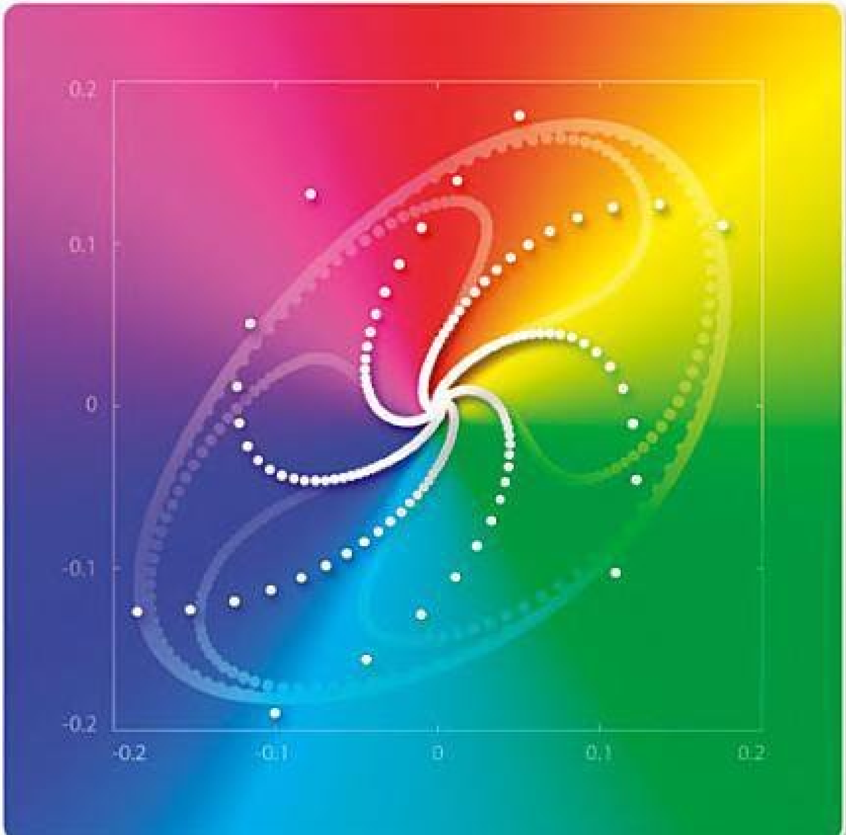
The Method of Normal Forms

Second, Updated and Enlarged Edition



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To my youngest son Nader

Contents

Preface

This book gives an introductory treatment of the method of normal forms. This technique has its application in many branches of engineering, physics, and applied mathematics. Approximation techniques such as these are important for people working with dynamical problems and are a valuable tool they should have in their tool box.

The exposition is largely by means of examples. The readers need not understand the physical bases of the examples used to describe the techniques. However, it is assumed that they have a knowledge of basic calculus as well as the elementary properties of ordinary differential equations. For most of the examples, the results obtained with the method of normal forms are shown to be equivalent to those obtained with other perturbation methods, such as the methods of multiple scales and averaging. As such, new sections are added treating some of the examples with these methods. Moreover, exercises are added to most chapters.

Because the normal forms of maps and differential equations are very useful in bifurcation analysis, I added in this edition three chapters dealing with the normal forms and bifurcations of maps, continuous systems, and retarded systems. The normal forms of continuous systems are constructed using the method of multiple scales, a combination of center-manifold reduction and the method of normal forms, and the new method of projection, which is developed first in this edition. Also, the normal forms of retarded systems are constructed using center-manifold reduction and the method of multiple scales. In the center-manifold reduction, we represent the retarded equations as operator differential equations, decompose the solution space of their linearized form into stable and center subspaces, define an inner product, determine the adjoint of the operator equations, calculate the center manifold, carry out details of the projection using the adjoint of the center subspace, and finally calculate the normal form on the center manifold.

I am very much indebted to my late parents, Hasan and Khadrah, who in spite of their lack of formal education insisted that all their sons obtain the highest degrees. If it were not for their incredible foresight on the value of an education even under the most severe conditions, I would not have finished secondary school. This book and its second edition would not have been written without the patience and continuous encouragement of my wife, Samirah.

Blacksburg, VA, December 2010 *Ali Hasan Nayfeh*

Introduction

The method of normal forms dates back to the days of Euler, Delaunay, Poincaré, Dulac, and Birkhoff. Moreover, the concept of using coordinate transformations to simplify mathematical problems involving algebraic, ordinary differential, partial differential, integral, and integro-differential equations has been used for a long time, as illustrated by the following examples.

As a first example, we consider Bessel's equation of order one-half; that is,

$$x^2 u'' + x u' + \left(x^2 - \tfrac{1}{4}\right) u = 0$$

Using the transformation $x^{-1/2} v(x)$, we transform this equation into the simple equation

$$v'' + v = 0$$

whose solution is

$$v = c_1 \cos x + c_2 \sin x$$

where c_1 and c_2 are arbitrary constants. Hence, Bessel's function of order one-half $J_{1/2}(x)$ is given by

$$J_{1/2}(x) = x^{-1/2} \left(c_1 \cos x + c_2 \sin x\right)$$

As a second example, we consider the vibrations of an n degree-of-freedom system governed by the following set of n coupled, linear equations of motion:

$$\ddot{\boldsymbol{x}} + K\boldsymbol{x} = 0$$

where $\boldsymbol{x}$ is a column vector of length n and K is an $n \times n$ constant symmetric matrix. Using the transformation $\boldsymbol{x} = P\boldsymbol{v}$, we obtain

$$\ddot{\boldsymbol{v}} + P^{-1} K P \boldsymbol{v} = 0$$

Assuming the eigenvalues $\lambda_1, \lambda_2, \ldots,$ and λ_n of K to be distinct and choosing the columns of P to be the orthonormal eigenvectors of K, we find that $P^{-1} K P$ is a

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diagonal matrix Λ with entries $\lambda_1, \lambda_2, \ldots, \lambda_n$. Hence, the system of equations can be written as

$$\ddot{\boldsymbol{v}} + \Lambda \boldsymbol{v} = 0$$

or in the decoupled form

$$\ddot{v}_i + \lambda_i v_i = 0 \quad \text{and} \quad i = 1, 2, \ldots, n$$

which is called the normal-modal form of

$$\ddot{\boldsymbol{x}} + K\boldsymbol{x} = 0$$

As a third example, we consider the system

$$\begin{aligned} \dot{a} &= -\mu a - \tfrac{1}{2} a \sin 2\beta \\ a\dot{\beta} &= -\tfrac{1}{2}\sigma a - \tfrac{1}{2} a \cos 2\beta \end{aligned}$$

where μ and σ are constants, which describes the time variation of the amplitude and phase of a parametrically excited linear oscillator in the case of a principal parametric resonance (Nayfeh and Mook, 1979). This system is nonlinear and its solution is not apparent. However, using the nonlinear transformation $x = a \cos \beta$ and $y = a \sin \beta$, we transform the nonlinear system into the following linear system:

$$\begin{aligned} \dot{x} &= -\mu x + \tfrac{1}{2} (\sigma - 1)\, y \\ \dot{y} &= -\mu y - \tfrac{1}{2} (\sigma + 1)\, x \end{aligned}$$

whose closed-form solution is readily obtainable.

As a fourth example, we consider the nonlinear system

$$\begin{aligned} \dot{x} &= y + (ax + by)\left(x^2 + y^2\right) \\ \dot{y} &= -x + (ay - bx)\left(x^2 + y^2\right) \end{aligned}$$

where a and b are constants, which describes the motion near a Hopf bifurcation point (Marsden and McCracken, 1976; Wiggins, 1990), as described in Section 4.4.5. Again the solution of this system is not apparent. However, using the nonlinear transformation $x = r \cos \beta$ and $y = -r \sin \beta$, we transform the system into

$$\begin{aligned} \dot{r} &= ar^3 \\ \dot{\beta} &= 1 + br^2 \end{aligned}$$

whose closed-form solution is readily obtainable.

As a fifth example, we consider the linear partial differential equation

$$u_{tt} - c^2 u_{xx} = f'(x - ct)\, f''(x - ct)$$

where f is a known twice differential function, the prime denotes the derivative with respect to the argument $(x - ct)$, and subscripts denote partial derivatives. The general solution of this equation can be readily obtained if we express the independent variables x and t in terms of the characteristics

$$\xi = x - ct \quad \text{and} \quad \eta = x + ct$$

Thus, this partial differential equation is transformed into

$$-4c^2 u_{\xi\eta} = f'(\xi) f''(\xi)$$

whose general solution is

$$u = -\frac{1}{8c^2} f'^2(\xi)\eta + g(\xi) + h(\eta)$$

where $g(\xi)$ and $h(\eta)$ are general functions of ξ and η.

As a sixth example, we consider the nonlinear partial differential equation

$$u_t + uu_x = \nu u_{xx}$$

where ν is a constant, which is known as Burger's equation (Whitham, 1974). Replacing u with ψ_x and integrating the result once yields

$$\psi_t + \tfrac{1}{2}\psi_x^2 = \nu\psi_{xx}$$

Then, using the nonlinear transformation $\psi = -2\nu \ln(\phi)$, Hopf (1950) and Cole (1951) transformed the nonlinear equation into the linear heat transfer equation

$$\phi_t = \nu\phi_{xx}$$

which can be solved much more easily than the original nonlinear equation.

As a seventh example, we consider the steady, incompressible, high-Reynolds number flow over a flat plate aligned with the oncoming uniform stream. The boundary layer approximation to the stream function $\psi(x, y)$ is governed by Van Dyke (1964)

$$\begin{aligned}
&\psi_{yyy} + \psi_x\psi_{yy} - \psi_y\psi_{xy} = 0\\
&\psi(x, 0) = 0\\
&\psi_y(x, 0) = 0 \quad \text{and} \quad 0 < x < \infty\\
&\psi_y(x, \infty) = 1
\end{aligned}$$

This nonlinear partial differential equation can be reduced to an ordinary differential equation by using the similarity transformation

$$\psi(x, y) = \sqrt{2x}\, f(\eta)\,, \quad \eta = y/\sqrt{2x}$$

With this transformation, the boundary layer problem becomes

$$f''' + ff'' = 0\,, \quad f(0) = 0\,, \quad f'(0) = 0\,, \quad f'(\infty) = 1$$

which is the Blasius problem.

In the preceding examples, transformations were introduced to transform a difficult problem into a more readily solvable problem. Next, we consider cases in which a transformation is used to transform the problem into a new "approximate" problem for which the exact solution can be readily obtained. Specifically, we consider the Liouville equation

$$y'' + \lambda^2 q(x) y = 0 \quad \text{when} \quad \lambda \gg 1$$

where λ is a constant, $q(x)$ is a known function, and the prime denotes the derivative with respect to x. To determine an approximate solution of this equation when $\lambda \gg 1$, we transform both of the dependent and independent variables as

$$z = \phi(x) \quad \text{and} \quad v(z) = \psi(x) y(x)$$

With this transformation, the Liouville equation becomes

$$\frac{d^2 v}{dz^2} + \frac{1}{\phi'^2}\left(\phi'' - \frac{2\phi'\psi'}{\psi}\right)\frac{dv}{dz} + \left(\frac{\lambda^2 q}{\phi'^2} - \frac{\psi''}{\psi\phi'^2} + \frac{2\psi'^2}{\psi^2\phi'^2}\right) v = 0$$

We choose ϕ and ψ so that the dominant part of the transformed equation has the simplest possible form and, at the same time, has solutions that have qualitatively the same behavior as the solutions of the original equations. In other words, we have to insist on the transformation being regular everywhere in the interval of interest. To this end, we force the coefficient of dv/dz to be zero; that is,

$$\phi'' - \frac{2\phi'\psi'}{\psi} = 0$$

Hence, $\psi = \sqrt{\phi'}$. In order that the transformation be regular, ψ must be regular and have no zeros in the interval of interest. Then, because $\psi = \sqrt{\phi'}$, ϕ' must be regular and have no zeros in the interval of interest. Consequently, we set

$$\lambda^2 q = \phi'^2 \xi(z)$$

so that the transformed equation becomes

$$\frac{d^2 v}{dz^2} + \xi(z) v = -\delta v$$

and choose the simplest possible function $\xi(z)$ that yields a nonsingular transformation. In order that ϕ' be regular and have no zeros in the interval of interest, $\xi(z)$ must have the same number, type, and order of singularities and zeros as q.

For example, when $q > 0$ everywhere in the interval of interest, the solutions of the original equation are oscillatory, and hence ϕ and ψ must be chosen so that the dominant part of the transformed equation is

$$\frac{d^2 v}{dz^2} + v = 0$$

which is the simplest possible equation with oscillatory solutions. When $q < 0$ everywhere in the interval of interest, one of the solutions of the original equation grows exponentially with x and the other decays exponentially with x. Hence, ϕ and ψ must be chosen so that the dominant part of the transformed equation is

$$\frac{d^2 v}{dz^2} - v = 0$$

which is the simplest possible equation with exponentially growing and decaying solutions.

When q changes sign once in the interval of interest, the solutions of the original equation are oscillatory on one side of the sign change and exponentially growing and decaying on the other side. For example, if $q = 1 - x^3$, the solutions of the original equation are oscillatory for $x < 1$ and exponential for $x > 1$. Hence, ϕ and ψ must be chosen so that the dominant part of the transformed equation has solutions whose behavior changes from oscillatory to exponentially growing and decaying at a given point. The simplest possible equation with these properties is the Airy equation

$$\frac{d^2 v}{dz^2} - zv = 0$$

When $z > 0$ the solutions of this equation are growing and decaying with z, and when $z < 0$ they are oscillatory. In other words, if $q(x)$ is regular and has only a simple zero (simple turning point) such as $1 - x^3$, then $\xi(z)$ must be chosen to be regular and have only a simple zero. The simplest possible function that satisfies these requirements is $\xi(z) = z$. If $q(x)$ is regular and has only a double zero at a point in the interval of interest (i.e., turning point of order 2), $\xi(z)$ must be chosen to be regular and have only a double zero. The simplest possible function satisfying these requirements is $\xi(z) = z^2$. If $q(x)$ is regular and has only a zero of order n (i.e., turning point of order n), $\xi(z)$ must be chosen to be z^n. If $q(x)$ has two zeros at $x = a$ and b, where $b > 1$, of order m and n, then one uses

$$\xi(z) = z^m (1 - z)^n$$

In analyzing oscillations of a weakly nonlinear system, the method of variation of parameters is usually used to transform the equations governing these oscillations into the standard form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}; \epsilon) = \sum_{m=0}^{\infty} \frac{\epsilon^m}{m!} \boldsymbol{f}_m(\boldsymbol{x})$$

where

$$\boldsymbol{f}_m(\boldsymbol{x}) = \left. \frac{\partial^m \boldsymbol{f}}{\partial \epsilon^m} \right|_{\epsilon=0}$$

Here $\boldsymbol{x}$ and $\boldsymbol{f}$ are vectors with N components. The vector $\boldsymbol{x}$ may represent, for example, the amplitudes and phases of the system. If we denote the components of the vector $\boldsymbol{f}_m$ by f_{mn}, then a component x_k of the vector $\boldsymbol{x}$ is said to be a rapidly rotating phase if $\boldsymbol{f}_{0k} \neq 0$.

To analyze this standard system, we introduce a near-identity transformation

$$\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{y};\epsilon) = \boldsymbol{y} + \epsilon \boldsymbol{X}_1(\boldsymbol{y}) + \epsilon^2 \boldsymbol{X}_2(\boldsymbol{y}) + \cdots$$

from $\boldsymbol{x}$ to $\boldsymbol{y}$ such that the system is transformed into

$$\dot{\boldsymbol{y}} = \boldsymbol{g}(\boldsymbol{y};\epsilon) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \boldsymbol{g}_n(\boldsymbol{y})$$

where the $\boldsymbol{g}_n$ contain long-period terms only. Using the generalized method of averaging (Nayfeh, 1973), one determines the $\boldsymbol{X}_n$ and $\boldsymbol{g}_n$ by substituting the transformation into the standard system and separating the short- and long-period terms assuming that the $\boldsymbol{X}_n$ contain short-period terms only.

Alternatively, we can define the transformation $\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{y};\epsilon)$ as the solution of the N differential equations

$$\frac{d\boldsymbol{x}}{d\epsilon} = \boldsymbol{W}(\boldsymbol{x};\epsilon)\,, \quad \boldsymbol{x}(\epsilon = 0) = \boldsymbol{y}$$

The vector $\boldsymbol{W}$ is called the generating vector. This equation generates the so-called Lie transforms (Kamel, 1970), which are invertible because they are close to the identity. It seems at first that we are going in circles because we are proposing to simplify the original system of differential equations by solving a system of N differential equations. This is not the case, because we are interested in the solution of the original system for large t, whereas we need the solution of the transformation for small ϵ, which is a significant simplification.

These examples clearly show that linear and nonlinear coordinate transformations can be used to simplify linear and nonlinear problems. A powerful method for systematically constructing these transformations is the method of normal forms. The basic idea underlying the method of normal forms is the use of "local" coordinate transformations to "simplify" the equations describing the dynamics of the system under consideration. In other words, with the method of normal forms, one seeks a near-identity coordinate transformation in which the dynamical system takes the "simplest" or so-called normal form. The transformations are generated in a neighborhood of a known solution, such as a fixed point (constant, stationary, or equilibrium solution) or a periodic orbit (limit cycle) of a system. In this text, the normalization is usually carried out with respect to a perturbation parameter.

1
SDOF Autonomous Systems

1.1
Introduction

In this chapter, we describe the method of normal forms using single-degree-of-freedom (SDOF) autonomous systems that can be modeled by the following second-order nonlinear ordinary differential equation:

$$\ddot{u} + \omega^2 u = f(u, \dot{u}) \tag{1.1}$$

where $f(u, \dot{u})$ can be developed in a power series in terms of u and $\dot{u}$. In what follows, we will refer to $\dot{u} + \omega^2 u = 0$ as the *unperturbed system* and (1.1) as the *perturbed system*. We assume that (1.1) has an equilibrium at $u = 0$ and $\dot{u} = 0$. Equation 1.1 can be cast as a system of two first-order equations by letting

$$x_1 = u \quad \text{and} \quad x_2 = \dot{u} \tag{1.2}$$

The result is

$$\dot{x}_1 = x_2 \tag{1.3}$$

$$\dot{x}_2 = -\omega^2 x_1 + f(x_1, x_2) \tag{1.4}$$

It is clear that the unperturbed system

$$\dot{x}_1 = x_2 \quad \text{and} \quad \dot{x}_2 = -\omega^2 x_1$$

has a simple pair of purely imaginary eigenvalues $\pm i\omega$.

The main idea underlying the method of normal forms is to introduce a near-identify transformation

$$x_1 = y_1 + h_1(y_1, y_2) \tag{1.5a}$$

$$x_2 = y_2 + h_2(y_1, y_2) \tag{1.5b}$$

from (x_1, x_2) to (y_1, y_2) into (1.3) and (1.4) to produce the simplest possible equations (the so-called normal form). We call the transformation (1.5) near-identity

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because $x_1(t) - y_1(t)$ and $x_2(t) - y_2(t)$ are small; that is, $o(x_1(t), x_2(t))$. This procedure is also called *normalization*. To this end, we substitute (1.5) into (1.3) and (1.4) and obtain

$$\dot{y}_1 = y_2 + h_2 - \frac{\partial h_1}{\partial y_1}\dot{y}_1 - \frac{\partial h_1}{\partial y_2}\dot{y}_2 \tag{1.6a}$$

$$\dot{y}_2 = -\omega^2 y_1 - \omega^2 h_1 + f(y_1 + h_1, y_2 + h_2) - \frac{\partial h_2}{\partial y_1}\dot{y}_1 - \frac{\partial h_2}{\partial y_2}\dot{y}_2 \tag{1.6b}$$

Then, we choose h_1 and h_2 such that (1.6) assume their simplest form. This task is accomplished in steps. If one decomposes $f(x_1, x_2)$ as

$$f(x_1, x_2) = \sum_{n=1}^{N} f_n(x_1, x_2) \tag{1.7}$$

where f_n is a polynomial of degree n in x_1 and x_2, then one chooses h_1 and h_2 to simplify the terms resulting from the lowest-order polynomial $f_m(x_1, x_2)$, where $m \geq 2$, in $f(x_1, x_2)$. In the next step, one chooses a second near-identity transformation to simplify the polynomial terms of degree $m + 1$, and so on.

It turns out that, because the unperturbed system (1.3) and (1.4) represents an oscillator, the governing equations can conveniently be expressed as a single complex-valued equation. To this end, we follow steps similar to those used in the method of variation of parameters (Nayfeh, 1981). When $f \equiv 0$, the solution of (1.1) can be expressed as

$$u = Be^{i\omega t} + \bar{B}e^{-i\omega t} \tag{1.8}$$

where B is a constant and $\bar{B}$ is the complex conjugate of B. Hence,

$$\dot{u} = i\omega\left(Be^{i\omega t} - \bar{B}e^{-i\omega t}\right) \tag{1.9}$$

When $f \neq 0$, we continue to represent the solution of (1.1) as in (1.8) subject to the constraint (1.9) but with time-varying rather than constant B. Next, we replace $Be^{i\omega t}$ with $\zeta(t)$ and rewrite (1.8) and (1.9) as

$$u = \zeta(t) + \bar{\zeta}(t) \quad \text{and} \quad \dot{u} = i\omega\left[\zeta(t) - \bar{\zeta}(t)\right] \tag{1.10}$$

Hence, solving for ζ and $\bar{\zeta}$, we obtain

$$\zeta = \frac{1}{2}\left(u - \frac{i}{\omega}\dot{u}\right) \quad \text{and} \quad \bar{\zeta} = \frac{1}{2}\left(u + \frac{i}{\omega}\dot{u}\right) \tag{1.11}$$

Differentiating (1.11) with respect to t yields

$$\dot{\zeta} = \frac{1}{2}\left(\dot{u} - \frac{i}{\omega}\ddot{u}\right) = \frac{1}{2}\left(\dot{u} + i\omega u - \frac{i}{\omega}f\right) \tag{1.12}$$

on account of (1.1). Hence,

$$\dot{\zeta} = \frac{1}{2}i\omega\left(u - \frac{i}{\omega}\dot{u}\right) - \frac{i}{2\omega}f(u, \dot{u}) \tag{1.13}$$

which, upon using (1.10), becomes

$$\dot{\zeta} = i\omega\zeta - \frac{i}{2\omega} f\left[\zeta + \bar{\zeta}, i\omega\left(\zeta - \bar{\zeta}\right)\right] \tag{1.14}$$

Next, we consider different polynomial forms for *f*.

1.2 Duffing Equation

The Duffing equation is

$$\ddot{u} + \omega^2 u = \alpha u^3$$

so that, in this case, $f = \alpha u^3$ and (1.14) becomes

$$\dot{\zeta} = i\omega\zeta - \frac{i\alpha}{2\omega}\left(\zeta + \bar{\zeta}\right)^3 \tag{1.15}$$

We introduce a near-identity transformation from ζ to η in the form

$$\zeta = \eta + h\left(\eta, \bar{\eta}\right) \tag{1.16}$$

and obtain

$$\dot{\eta} = i\omega\eta + i\omega h - \frac{\partial h}{\partial \eta}\dot{\eta} - \frac{\partial h}{\partial \bar{\eta}}\dot{\bar{\eta}} - \frac{i\alpha}{2\omega}\left(\eta + h + \bar{\eta} + \bar{h}\right)^3 \tag{1.17}$$

Because the nonlinearity is cubic, we assume that h is third order in η and $\bar{\eta}$; that is,

$$h = \Lambda_1\eta^3 + \Lambda_2\eta^2\bar{\eta} + \Lambda_3\eta\bar{\eta}^2 + \Lambda_4\bar{\eta}^3 \tag{1.18}$$

and choose the Λ_i so that (1.17) takes the simplest possible (normal) form.

In the first step, we eliminate $\dot{\eta}$ and $\dot{\bar{\eta}}$ from the right-hand side of (1.17). This task is accomplished by iteration. To the first approximation, it follows from (1.17) that

$$\dot{\eta} = i\omega\eta \quad \text{and} \quad \dot{\bar{\eta}} = -i\omega\bar{\eta} \tag{1.19}$$

Next, we replace $\dot{\eta}$ and $\dot{\bar{\eta}}$ on the right-hand side of (1.17) using (1.19), use (1.18), keep up to third-order terms, and obtain

$$\begin{aligned}\dot{\eta} = {} & i\omega\eta - i\omega\left(2\Lambda_1 + \frac{\alpha}{2\omega^2}\right)\eta^3 - \frac{3i\alpha}{2\omega}\eta^2\bar{\eta} + i\omega\left(2\Lambda_3 - \frac{3\alpha}{2\omega^2}\right)\eta\bar{\eta}^2 \\ & + i\omega\left(4\Lambda_4 - \frac{\alpha}{2\omega^2}\right)\bar{\eta}^3\end{aligned} \tag{1.20}$$

Next, we choose Λ_1, Λ_3, and Λ_4 to eliminate the terms involving η^3, $\eta\bar{\eta}^2$, and $\bar{\eta}^3$; that is,

$$\Lambda_1 = -\frac{\alpha}{4\omega^2}\,, \quad \Lambda_3 = \frac{3\alpha}{4\omega^2}\,, \quad \Lambda_4 = \frac{\alpha}{8\omega^2} \tag{1.21}$$

However, because Λ_2 does not appear in (1.20), the term involving $\eta^2\bar{\eta}$ cannot be eliminated; it is called a *resonance term*. Consequently, to the second approximation, the simplest possible form for $\dot{\eta}$ is

$$\dot{\eta} = i\omega\eta - \frac{3i\alpha}{2\omega}\eta^2\bar{\eta} \tag{1.22}$$

To show that $\eta^2\bar{\eta}$ is a resonance term, we find a solution for (1.22) by iteration. To the first approximation, $\eta = Ae^{i\omega t}$, where A is a constant. Then, (1.22) becomes

$$\dot{\eta} = i\omega\eta - \frac{3i\alpha}{2\omega}A^2\bar{A}e^{i\omega t}$$

whose solution can be written as

$$\eta = Ae^{i\omega t} - \frac{3\alpha}{2\omega}A^2\bar{A}te^{i\omega t} \tag{1.23a}$$

It is clear that this expansion, which is also a straightforward expansion, is nonuniform for large t because of the presence of a secular term created by $\eta^2\bar{\eta}$. Alternatively, we can demonstrate that the term $\zeta^2\bar{\zeta}$ is a *resonance term* in the original equation (1.15). To the first approximation, we neglect the nonlinear term in (1.15) and find that $\zeta = Ae^{i\omega t}$. Then, to the second approximation, (1.15) becomes

$$\dot{\zeta} = i\omega\zeta - \frac{i\alpha}{2\omega}\left(A^3e^{3i\omega t} + 3A^2\bar{A}e^{i\omega t} + 3A\bar{A}^2e^{-i\omega t} + \bar{A}^3e^{-3i\omega t}\right)$$

whose solution can be written as

$$\zeta = Ae^{i\omega t} - \frac{\alpha}{4\omega^2}A^3e^{3i\omega t} - \frac{3i\alpha}{2\omega}A^2\bar{A}te^{i\omega t} + \frac{3\alpha}{4\omega^2}A\bar{A}^2e^{-i\omega t} + \frac{\alpha}{8\omega^2}\bar{A}^3e^{-3i\omega t} \tag{1.23b}$$

It is clear that this expansion is nonuniform because of the presence of a secular term created by $\zeta^2\bar{\zeta}$. The other three terms proportional to $A^3e^{3i\omega t}$, $A\bar{A}^2e^{-i\omega t}$, and $\bar{A}^3e^{-3i\omega t}$ created by ζ^3, $\zeta\bar{\zeta}^2$, and $\bar{\zeta}^3$ do not produce secular terms and hence they are *nonresonance*. Consequently, one can choose a near-identity transformation to eliminate them.

As a second alternative, starting with the original equation (1.15), we break the nonlinear part $f(\zeta, \bar{\zeta})$ into two parts as

$$f(\zeta, \bar{\zeta}) = f_1(\zeta, \bar{\zeta}) + f_2(\zeta, \bar{\zeta})$$

where

$$e^{-i\omega t}f_1\left(e^{i\omega t}, e^{-i\omega t}\right)$$

is *time invariant*, whereas

$$e^{-i\omega t} f_2\left(e^{i\omega t}, e^{-i\omega t}\right)$$

is not time invariant. In the present case,

$$f = \left(\zeta + \bar{\zeta}\right)^3, \quad f_1 = 3\zeta^2\bar{\zeta}, \quad f_2 = \zeta^3 + 3\zeta\bar{\zeta}^2 + \bar{\zeta}^3$$

Thus,

$$e^{-i\omega t} f_1\left(e^{i\omega t}, e^{-i\omega t}\right) = e^{-i\omega t}\left(3e^{2i\omega t}e^{-i\omega t}\right) = 3$$

which is time invariant, whereas

$$e^{-i\omega t} f_2\left(e^{i\omega t}, e^{-i\omega t}\right) = e^{2i\omega t} + 3e^{-2i\omega t} + e^{-4i\omega t}$$

which does not contain any time-invariant terms.

Substituting (1.16) and (1.18) into (1.10), using (1.21), and setting $\Lambda_2 = 0$ because it is arbitrary yields

$$u = \eta + \bar{\eta} - \frac{\alpha}{8\omega^2}\left(\eta^3 + \bar{\eta}^3\right) + \frac{3\alpha}{4\omega^2}\left(\eta\bar{\eta}^2 + \eta^2\bar{\eta}\right) \tag{1.24}$$

where η is given by (1.22). Next, we separate the fast from the slow variations in η by introducing the transformation

$$\eta = A(t)e^{i\omega t}$$

where ω is the natural frequency of the system and A is a function of time, into (1.22) and (1.24) and obtain

$$\dot{A} = -\frac{3i\alpha}{2\omega}A^2\bar{A} \tag{1.25}$$

$$u = Ae^{i\omega t} + \bar{A}e^{-i\omega t} - \frac{\alpha}{8\omega^2}\left(A^3e^{3i\omega t} + \bar{A}^3e^{-3i\omega t}\right) + \frac{3\alpha}{4\omega^2}\left(A^2\bar{A}e^{i\omega t} + \bar{A}^2Ae^{-i\omega t}\right) + \cdots \tag{1.26}$$

Expressing A in the polar form

$$A = \tfrac{1}{2}ae^{i\beta} \tag{1.27}$$

where a and β are functions of t, we rewrite (1.26) as

$$u = \left(a + \frac{3\alpha}{16\omega^2}a^3\right)\cos(\omega t + \beta) - \frac{\alpha a^3}{32\omega^2}\cos(3\omega t + 3\beta) + \cdots \tag{1.28}$$

Substituting (1.27) into (1.25) and separating real and imaginary parts, we have

$$\dot{a} = 0 \tag{1.29}$$

$$a\dot{\beta} = -\frac{3\alpha}{8\omega}a^3 \tag{1.30}$$

In determining the normal form (1.22), we had to use an ordering scheme to indicate the relative magnitudes of the different terms in (1.15). We based the ordering scheme on the fact that ζ and $\bar{\zeta}$ are small and hence $\zeta^3, \zeta^2\bar{\zeta}, \zeta\bar{\zeta}^2$, and $\bar{\zeta}^3$ are much smaller than ζ and $\bar{\zeta}$. In other words, we based the ordering scheme on the degree of the terms. This worked well in this example, but there are many physical systems where the ordering does not follow from the degree of the polynomial but from the presence of certain parameters in their models. We consider such an example in the next section.

Next, we treat (1.15) by using the method of multiple scales. To this end, we introduce a small nondimensional parameter ϵ as a bookkeeping device and rewrite (1.15) as

$$\dot{\zeta} = i\omega\zeta - \frac{i\epsilon\alpha}{2\omega}\left(\zeta + \bar{\zeta}\right)^3 \tag{1.31}$$

Then, we seek an approximate solution of (1.31) in the form

$$\zeta(t;\epsilon) = \zeta_0\left(T_0, T_1\right) + \epsilon\zeta_1\left(T_0, T_1\right) + \cdots \tag{1.32}$$

where $T_n = \epsilon^n t$ and

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} + \epsilon\frac{\partial}{\partial T_1} + \cdots = D_0 + \epsilon D_1 + \cdots \tag{1.33}$$

Substituting (1.32) and (1.33) into (1.31) and equating coefficients of like powers of ϵ yields

Order (ϵ^0)

$$D_0\zeta_0 - i\omega\zeta_0 = 0 \tag{1.34}$$

Order (ϵ)

$$D_0\zeta_1 - i\omega\zeta_1 = -D_1\zeta_0 - \frac{i\alpha}{2\omega}\left(\zeta_0 + \bar{\zeta}_0\right)^3 \tag{1.35}$$

The solution of (1.34) can be expressed as

$$\zeta_0 = A(T_1)e^{i\omega T_0} \tag{1.36}$$

Then, (1.35) becomes

$$\begin{aligned} D_0\zeta_1 - i\omega\zeta_1 = -A'e^{i\omega T_0} - \frac{i\alpha}{2\omega}\big(&A^3e^{3i\omega T_0} + 3A^2\bar{A}e^{i\omega T_0} \\ &+ 3A\bar{A}^2e^{-i\omega T_0} + \bar{A}^3e^{-3i\omega T_0}\big) \end{aligned} \tag{1.37}$$

Eliminating the terms that lead to secular terms from (1.37), we have

$$A' = -\frac{3i\alpha}{2\omega}A^2\bar{A} \tag{1.38}$$

Then, a particular solution of (1.37) can be expressed as

$$\zeta_1 = -\frac{\alpha}{4\omega^2} A^3 e^{3i\omega T_0} + \frac{3\alpha}{4\omega^2} A\bar{A}^2 e^{-i\omega T_0} + \frac{\alpha}{8\omega^2} \bar{A}^3 e^{-3i\omega T_0} \tag{1.39}$$

Substituting (1.36) and (1.39) into (1.10), we obtain

$$u = Ae^{i\omega t} + \bar{A}e^{-i\omega t} - \frac{\epsilon\alpha}{8\omega^2}\left(A^3 e^{3i\omega t} + \bar{A}^3 e^{-3i\omega t}\right) + \frac{3\epsilon\alpha}{4\omega^2}\left(A^2\bar{A}e^{i\omega t} + A\bar{A}^2 e^{-i\omega t}\right) + \cdots \tag{1.40}$$

Equations 1.38–1.40 are in full agreement with (1.25) and (1.26) obtained with the method of normal forms because $T_1 = \epsilon t$ and ϵ can be set equal to unity.

1.3 Rayleigh Equation

The Rayleigh equation is

$$\ddot{u} + \omega^2 u = \epsilon\left(\dot{u} - \tfrac{1}{3}\dot{u}^3\right) \tag{1.41}$$

where ϵ is a small, positive nondimensional parameter. Here

$$f = \epsilon\left(\dot{u} - \tfrac{1}{3}\dot{u}^3\right)$$

and (1.14) becomes

$$\dot{\zeta} = i\omega\zeta + \tfrac{1}{2}\epsilon\left[\zeta - \bar{\zeta} + \tfrac{1}{3}\omega^2\left(\zeta - \bar{\zeta}\right)^3\right] \tag{1.42}$$

In this example, the ordering is not based on the degree of the polynomial, but on the small nondimensional parameter ϵ. Normalization is carried out in terms of the small parameter ϵ. In fact, the perturbation contains linear as well as cubic terms.

Using the transformation (1.16), we rewrite (1.42) as

$$\dot{\eta} = i\omega\eta + i\omega h - \frac{\partial h}{\partial\eta}\dot{\eta} - \frac{\partial h}{\partial\bar{\eta}}\dot{\bar{\eta}} + \frac{1}{2}\epsilon\left[\eta - \bar{\eta} + h - \bar{h} + \frac{1}{3}\omega^2\left(\eta - \bar{\eta} + h - \bar{h}\right)^3\right] \tag{1.43}$$

Because the perturbation in (1.43) involves linear and cubic terms, we express h in the form

$$h = \epsilon\left(\Delta_1\eta + \Delta_2\bar{\eta} + \Lambda_1\eta^3 + \Lambda_2\eta^2\bar{\eta} + \Lambda_3\eta\bar{\eta}^2 + \Lambda_4\bar{\eta}^3\right) \tag{1.44}$$

Moreover, to the first approximation, $\dot{\eta}$ and $\dot{\bar{\eta}}$ are given by (1.19). Then, substituting (1.19) and (1.44) into the right-hand side of (1.43) and keeping terms up to $O(\epsilon)$, we obtain

$$\dot{\eta} = i\omega\eta + 2i\epsilon\omega\left(\Delta_2 + \frac{i}{4\omega}\right)\bar{\eta} + \frac{1}{2}\epsilon\eta - i\epsilon\omega\left(2\Lambda_1 + \frac{1}{6}i\omega\right)\eta^3$$
$$- \frac{1}{2}\epsilon\omega^2\eta^2\bar{\eta} + i\epsilon\omega\left(2\Lambda_3 - \frac{1}{2}i\omega\right)\eta\bar{\eta}^2 + i\epsilon\omega\left(4\Lambda_4 + \frac{1}{6}i\omega\right)\bar{\eta}^3 \tag{1.45}$$

We note that (1.45) is independent of Δ_1 and Λ_2 and hence they are arbitrary. Moreover, the terms proportional to $\epsilon\eta$ and $\epsilon\eta^2\bar{\eta}$ are resonance terms and hence cannot be eliminated from (1.45). Next, we choose $\Delta_2, \Lambda_1, \Lambda_3$, and Λ_4 to eliminate the terms involving $\bar{\eta}, \eta^3, \eta\bar{\eta}^2$, and $\bar{\eta}^3$, thereby producing the simplest possible equation for η. Thus, we have

$$\Delta_2 = -\frac{i}{4\omega}\,, \quad \Lambda_1 = -\frac{1}{12}i\omega\,, \quad \Lambda_3 = \frac{1}{4}i\omega\,, \quad \Lambda_4 = -\frac{1}{24}i\omega \tag{1.46}$$

With this choice, (1.45) takes the normal form

$$\dot{\eta} = i\omega\eta + \tfrac{1}{2}\epsilon\eta - \tfrac{1}{2}\epsilon\omega^2\eta^2\bar{\eta} \tag{1.47}$$

Again, in this case, we could have identified the resonance terms in (1.42) by one of the procedures described in Section 1.2. Because the solution of the unperturbed problem is proportional to $e^{i\omega t}$, the resonance terms in

$$f(\zeta, \bar{\zeta}) = \zeta - \bar{\zeta} + \tfrac{1}{3}\omega^2(\zeta - \bar{\zeta})^3$$

are the terms proportional to $e^{i\omega t}$ or the time-invariant terms in

$$e^{-i\omega t} f\left[e^{i\omega t} - e^{-i\omega t}, i\omega\left(e^{i\omega t} - e^{-i\omega t}\right)\right]$$

A simple calculation shows that the term $1/2\epsilon(\zeta - \omega^2\zeta^2\bar{\zeta})$ is the only resonance term. Hence, keeping only the resonance terms in (1.42), we have

$$\dot{\zeta} = i\omega\zeta + \tfrac{1}{2}\epsilon\left(\zeta - \omega^2\zeta^2\bar{\zeta}\right) + \cdots$$

which is formally equivalent to (1.47).

Next, we treat (1.42) with the method of multiple scales. To this end, we substitute (1.32) and (1.33) into (1.42), equate coefficients of equal powers of ϵ, and obtain

Order (ϵ^0)

$$D_0\zeta_0 - i\omega\zeta_0 = 0 \tag{1.48}$$

Order (ϵ)

$$D_0\zeta_1 - i\omega\zeta_1 = -D_1\zeta_0 + \tfrac{1}{2}\left[\zeta_0 - \bar{\zeta}_0 + \tfrac{1}{3}\omega^2\left(\zeta_0 - \bar{\zeta}_0\right)^3\right] \tag{1.49}$$

The solution of (1.48) can be expressed as

$$\zeta_0 = A(T_1)e^{i\omega T_0} \tag{1.50}$$

Then, (1.49) becomes

$$\begin{aligned} D_0\zeta_1 - i\omega\zeta_1 = &-A'e^{i\omega T_0} + \tfrac{1}{2}Ae^{i\omega T_0} - \tfrac{1}{2}\bar{A}e^{-i\omega T_0} + \tfrac{1}{6}\omega^2 A^3 e^{3i\omega T_0} \\ &- \tfrac{1}{2}\omega^2 A^2\bar{A}e^{i\omega T_0} + \tfrac{1}{2}\omega^2 A\bar{A}^2 e^{-i\omega T_0} - \tfrac{1}{6}\omega^2\bar{A}^3 e^{-3i\omega T_0} \end{aligned} \tag{1.51}$$

Eliminating the terms that lead to secular terms from (1.51), we have

$$A' = \tfrac{1}{2}A - \tfrac{1}{2}\omega^2 A^2\bar{A} \tag{1.52}$$

Letting $\eta = Ae^{i\omega t}$ in (1.47), we obtain (1.52) because $T_1 = \epsilon t$.

1.4 Duffing–Rayleigh–van der Pol Equation

The Duffing, Rayleigh, and van der Pol equations are special cases of

$$\ddot{u} + \omega^2 u = \epsilon\left(\mu\dot{u} + \alpha_1 u^3 + \alpha_2 u^2\dot{u} + \alpha_3 u\dot{u}^2 + \alpha_4\dot{u}^3\right) \tag{1.53}$$

so that

$$f = \epsilon\left(\mu\dot{u} + \alpha_1 u^3 + \alpha_2 u^2\dot{u} + \alpha_3 u\dot{u}^2 + \alpha_4\dot{u}^3\right)$$

and (1.14) becomes

$$\begin{aligned} \dot{\zeta} = i\omega\zeta - \frac{i\epsilon}{2\omega}\Big[&i\mu\omega\left(\zeta - \bar{\zeta}\right) + \alpha_1\left(\zeta + \bar{\zeta}\right)^3 + i\omega\alpha_2\left(\zeta + \bar{\zeta}\right)^2\left(\zeta - \bar{\zeta}\right) \\ &-\omega^2\alpha_3\left(\zeta + \bar{\zeta}\right)\left(\zeta - \bar{\zeta}\right)^2 - i\omega^3\alpha_4\left(\zeta - \bar{\zeta}\right)^3\Big] \end{aligned} \tag{1.54}$$

Using the transformation (1.16), where $h = O(\epsilon)$, we rewrite (1.54) as

$$\begin{aligned} \dot{\eta} = {} & i\omega\eta + i\omega h - \frac{\partial h}{\partial\eta}\dot{\eta} - \frac{\partial h}{\partial\bar{\eta}}\dot{\bar{\eta}} \\ & - \frac{i\epsilon}{2\omega}\left[i\mu\omega\left(\eta - \bar{\eta}\right) + i\omega\alpha_2\left(\eta + \bar{\eta}\right)^2\left(\eta - \bar{\eta}\right)\right. \\ & \left. +\alpha_1\left(\eta + \bar{\eta}\right)^3 - \omega^2\alpha_3\left(\eta + \bar{\eta}\right)\left(\eta - \bar{\eta}\right)^2 - i\omega^3\alpha_4\left(\eta - \bar{\eta}\right)^3\right] \end{aligned} \tag{1.55}$$

where terms of $O(\epsilon^2)$ and higher have been neglected.

Again, because the perturbation contains linear as well as third-order terms, h has the form (1.44). Moreover, to the first approximation, $\dot{\eta}$ and $\dot{\bar{\eta}}$ are given by

(1.19). Hence, substituting (1.19) and (1.44) into (1.55) yields

$$\begin{aligned}\dot{\eta} &= i\omega\eta + 2i\epsilon\omega\left(\Delta_2 + \frac{i\mu}{4\omega}\right)\bar{\eta} \\ &\quad - \frac{i\epsilon}{2\omega}\left(3\alpha_1 + i\omega\alpha_2 + \omega^2\alpha_3 + 3i\omega^3\alpha_4\right)\eta^2\bar{\eta} \\ &\quad + \frac{1}{2}\epsilon\mu\eta + i\epsilon\omega\left[-2\Lambda_1 - \frac{1}{2\omega^2}\left(\alpha_1 + i\omega\alpha_2 - \omega^2\alpha_3 - i\omega^3\alpha_4\right)\right]\eta^3 \\ &\quad + i\epsilon\omega\left[2\Lambda_3 - \frac{1}{2\omega^2}\left(3\alpha_1 - i\omega\alpha_2 + \omega^2\alpha_3 - 3i\omega^3\alpha_4\right)\right]\eta\bar{\eta}^2 \\ &\quad + i\epsilon\omega\left[4\Lambda_4 - \frac{1}{2\omega^2}\left(\alpha_1 - i\omega\alpha_2 - \omega^2\alpha_3 + i\omega^3\alpha_4\right)\right]\bar{\eta}^3 \end{aligned} \tag{1.56}$$

We note that Δ_1 and Λ_2 do not appear in (1.56) and hence they are arbitrary and the terms η and $\eta^2\bar{\eta}$ are resonance terms. To produce the simplest form for (1.56), we choose $\Delta_2, \Lambda_1, \Lambda_3$, and Λ_4 to eliminate the terms involving $\bar{\eta}, \eta^3, \eta\bar{\eta}^2$, and $\bar{\eta}^3$; that is,

$$\Delta_2 = -\frac{i\mu}{4\omega} \tag{1.57}$$

$$\Lambda_1 = -\frac{1}{4\omega^2}\left(\alpha_1 + i\omega\alpha_2 - \omega^2\alpha_3 - i\omega^3\alpha_4\right) \tag{1.58}$$

$$\Lambda_3 = \frac{1}{4\omega^2}\left(3\alpha_1 - i\omega\alpha_2 + \omega^2\alpha_3 - 3i\omega^3\alpha_4\right) \tag{1.59}$$

$$\Lambda_4 = \frac{1}{8\omega^2}\left(\alpha_1 - i\omega\alpha_2 - \omega^2\alpha_3 + i\omega^3\alpha_4\right) \tag{1.60}$$

With these choices, (1.56) assumes the simple form

$$\dot{\eta} = i\omega\eta + \frac{1}{2}\epsilon\mu\eta - \frac{i\epsilon}{2\omega}\left(3\alpha_1 + i\omega\alpha_2 + \omega^2\alpha_3 + 3i\omega^3\alpha_4\right)\eta^2\bar{\eta} \tag{1.61}$$

Again, we did not have to go through the lengthy algebra to arrive at the normal form (1.61). Because the solution of the unperturbed problem (1.54) is proportional to $e^{i\omega t}$, we could have replaced ζ with $e^{i\omega t}$ in the perturbation and identified the terms proportional to $e^{i\omega t}$. In this case, they are

$$\frac{1}{2}\epsilon\mu\zeta - \frac{i\epsilon}{2\omega}\left(3\alpha_1 + i\omega\alpha_2 + \omega^2\alpha_3 + 3i\omega^3\alpha_4\right)\zeta^2\bar{\zeta}$$

Hence, keeping only the resonance terms in (1.54), we obtain the normal form

$$\dot{\zeta} = i\omega\zeta + \frac{1}{2}\epsilon\mu\zeta - \frac{i\epsilon}{2\omega}\left(3\alpha_1 + i\omega\alpha_2 + \omega^2\alpha_3 + 3i\omega^3\alpha_4\right)\zeta^2\bar{\zeta}$$

which is formally equivalent to (1.61).

1.5 An Oscillator with Quadratic and Cubic Nonlinearities

We consider free oscillations of a single-degree-of-freedom system governed by

$$\ddot{u} + \omega^2 u + \delta u^2 + \alpha u^3 = 0 \tag{1.62}$$

To keep track of the different orders of magnitude, we use a nondimensional parameter ϵ that is the order of the amplitude of oscillations and hence rewrite (1.62) as

$$\ddot{u} + \omega^2 u + \epsilon \delta u^2 + \epsilon^2 \alpha u^3 = 0 \tag{1.63}$$

Thus, $f = -\epsilon \delta u^2 - \epsilon^2 \alpha u^3$ and (1.14) becomes

$$\dot{\zeta} = i\omega\zeta + \frac{i\epsilon\delta}{2\omega}\left(\zeta + \bar{\zeta}\right)^2 + \frac{i\epsilon^2\alpha}{2\omega}\left(\zeta + \bar{\zeta}\right)^3 \tag{1.64}$$

In the next section, we use two successive transformations to produce the normal form of (1.64). In Section 1.5.3, we use a single transformation to produce the same normal form, and in Section 1.5.2, we use the method of multiple scales to determine a second-order expansion of (1.64).

1.5.1 Successive Transformations

To simplify the $O(\epsilon)$ terms in (1.64), we introduce the near-identity transformation

$$\zeta = \eta + \epsilon h_1(\eta, \bar{\eta}) \tag{1.65}$$

and rewrite (1.64) as

$$\dot{\eta} = i\omega\eta + i\epsilon\omega h_1 - \epsilon\frac{\partial h_1}{\partial \eta}\dot{\eta} - \epsilon\frac{\partial h_1}{\partial \bar{\eta}}\dot{\bar{\eta}} + \frac{i\epsilon\delta}{2\omega}\left(\eta + \bar{\eta} + \epsilon h_1 + \epsilon \bar{h}_1\right)^2 + \frac{i\epsilon^2\alpha}{2\omega}\left(\eta + \bar{\eta}\right)^3 + \cdots \tag{1.66}$$

The form of the $O(\epsilon)$ terms suggests choosing h_1 in the form

$$h_1 = \Gamma_1\eta^2 + \Gamma_2\eta\bar{\eta} + \Gamma_3\bar{\eta}^2 \tag{1.67}$$

It follows from (1.66) that

$$\dot{\eta} = i\omega\eta + O(\epsilon) \quad \text{and} \quad \dot{\bar{\eta}} = -i\omega\bar{\eta} + O(\epsilon)$$

so that to $O(\epsilon)$ (1.66) becomes

$$\dot{\eta} = i\omega\eta + i\epsilon\omega\left(-\Gamma_1 + \frac{\delta}{2\omega^2}\right)\eta^2 + i\epsilon\omega\left(\Gamma_2 + \frac{\delta}{\omega^2}\right)\eta\bar{\eta} + i\epsilon\omega\left(3\Gamma_3 + \frac{\delta}{2\omega^2}\right)\bar{\eta}^2 + O(\epsilon^2) \tag{1.68}$$

The simplest possible form for (1.68) corresponds to the vanishing of the terms involving η^2, $\eta\bar{\eta}$, and $\bar{\eta}^2$; that is, choosing Γ_1, Γ_2, and Γ_3 to be

$$\Gamma_1 = \frac{\delta}{2\omega^2}\,, \quad \Gamma_2 = -\frac{\delta}{\omega^2}\,, \quad \Gamma_3 = -\frac{\delta}{6\omega^2} \tag{1.69}$$

Then, (1.68) reduces to

$$\dot{\eta} = i\omega\eta + O(\epsilon^2) \tag{1.70}$$

Substituting (1.67) into (1.66) and using (1.70) to eliminate $\dot{\eta}$ and $\dot{\bar{\eta}}$, we obtain

$$\dot{\eta} = i\omega\eta + \frac{i\epsilon^2}{2\omega}\left[\alpha\,(\eta+\bar{\eta})^3 + \frac{2\delta^2}{3\omega^2}\left(\eta^3+\bar{\eta}^3-5\eta^2\bar{\eta}-5\eta\bar{\eta}^2\right)\right] + \cdots \tag{1.71}$$

Next, we introduce a near-identity transformation from η to ξ in the form

$$\eta = \xi + \epsilon^2 h_2\left(\xi, \bar{\xi}\right) \tag{1.72}$$

and obtain

$$\begin{aligned}\dot{\xi} &= i\omega\xi + i\epsilon^2\omega h_2 - \epsilon^2\frac{\partial h_2}{\partial\xi}\dot{\xi} - \epsilon^2\frac{\partial h_2}{\partial\bar{\xi}}\dot{\bar{\xi}} \\ &\quad + \frac{i\epsilon^2}{2\omega}\left[\alpha\left(\xi+\bar{\xi}\right)^3 + \frac{2\delta^2}{3\omega^2}\left(\xi^3+\bar{\xi}^3-5\xi^2\bar{\xi}-5\xi\bar{\xi}^2\right)\right] + \cdots\end{aligned} \tag{1.73}$$

The form of the $O(\epsilon^2)$ terms suggests choosing h_2 in the form

$$h_2 = \Lambda_1\xi^3 + \Lambda_2\xi^2\bar{\xi} + \Lambda_3\xi\bar{\xi}^2 + \Lambda_4\bar{\xi}^3 \tag{1.74}$$

It follows from (1.73) that

$$\dot{\xi} = i\omega\xi + O(\epsilon^2) \quad \text{and} \quad \dot{\bar{\xi}} = -i\omega\bar{\xi} + O(\epsilon^2) \tag{1.75}$$

Therefore, substituting (1.74) and (1.75) into the right-hand side of (1.73) and keeping terms up to $O(\epsilon^2)$, we have

$$\begin{aligned}\dot{\xi} &= i\omega\xi + i\epsilon^2\omega\left(-2\Lambda_1 + \frac{\alpha}{2\omega^2} + \frac{\delta^2}{3\omega^4}\right)\xi^3 \\ &\quad + i\epsilon^2\omega\left(4\Lambda_4 + \frac{\alpha}{2\omega^2} + \frac{\delta^2}{3\omega^4}\right)\bar{\xi}^3 + \frac{i\epsilon^2}{2\omega}\left(3\alpha - \frac{10\delta^2}{3\omega^2}\right)\xi^2\bar{\xi} \\ &\quad + i\epsilon^2\omega\left(2\Lambda_3 + \frac{3\alpha}{2\omega^2} - \frac{5\delta^2}{3\omega^4}\right)\xi\bar{\xi}^2 + \cdots\end{aligned} \tag{1.76}$$

We note that (1.76) is independent of Λ_2 and hence it is arbitrary and the term $\xi^2\bar{\xi}$ is a resonance term. Equation 1.76 takes the simplest possible form if

$$\Lambda_1 = \frac{\alpha}{4\omega^2} + \frac{\delta^2}{6\omega^4}\,, \quad \Lambda_3 = -\frac{3\alpha}{4\omega^2} + \frac{5\delta^2}{6\omega^4}\,, \quad \Lambda_4 = -\frac{\alpha}{8\omega^2} - \frac{\delta^2}{12\omega^4} \tag{1.77}$$

Then, (1.76) becomes

$$\dot{\xi} = i\omega\xi + \frac{i\epsilon^2}{2\omega}\left(3\alpha - \frac{10\delta^2}{3\omega^2}\right)\xi^2\bar{\xi} + \cdots \tag{1.78}$$

Substituting (1.65) into (1.10), we have

$$u = \zeta + \bar{\zeta} = \eta + \bar{\eta} + \epsilon h_1(\eta, \bar{\eta}) + \epsilon \bar{h}_1(\eta, \bar{\eta}) \tag{1.79}$$

Then, substituting (1.72) into (1.79) yields

$$u = \xi + \bar{\xi} + \epsilon h_1(\xi, \bar{\xi}) + \epsilon \bar{h}_1(\xi, \bar{\xi}) + \cdots \tag{1.80}$$

Substituting for h_1 from (1.67) into (1.80) and using (1.69), we obtain

$$u = \xi + \bar{\xi} + \frac{\epsilon\delta}{3\omega^2}(\xi^2 - 6\xi\bar{\xi} + \bar{\xi}^2) + \cdots \tag{1.81}$$

Substituting the polar form

$$\xi = \tfrac{1}{2} a e^{i(\omega t + \beta)}$$

into (1.81), we find that

$$u = a\cos(\omega t + \beta) + \frac{\epsilon\delta a^2}{6\omega^2}[\cos(2\omega t + 2\beta) - 3] + \cdots \tag{1.82}$$

Substituting the polar form into (1.78) and separating real and imaginary parts, we have

$$\dot{a} = 0 \tag{1.83}$$

$$a\dot{\beta} = \epsilon^2 a^3\left(\frac{3\alpha}{8\omega} - \frac{5\delta^2}{12\omega^3}\right) \tag{1.84}$$

Equations 1.82–1.84 are in full agreement with those obtained by using the method of multiple scales, as shown in the next section.

1.5.2
The Method of Multiple Scales

Using the method of multiple scales, we seek a second-order uniform expansion of the solution of (1.64) in the form

$$\zeta(t;\epsilon) = \sum_{n=0}^{2} \epsilon^n \zeta_n(T_0, T_1, T_2) + \cdots \tag{1.85}$$

where $T_n = \epsilon^n t$. In terms of these scales, the time derivative becomes

$$\frac{d}{dt} = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \cdots, \quad D_n = \frac{\partial}{\partial T_n} \tag{1.86}$$

Substituting (1.85) and (1.86) into (1.64) and equating coefficients of like powers of ϵ, we obtain

Order (ϵ^0)

$$D_0\zeta_0 - i\omega\zeta_0 = 0 \tag{1.87}$$

Order (ϵ)

$$D_0\zeta_1 - i\omega\zeta_1 = -D_1\zeta_0 + \frac{i\delta}{2\omega}\left(\zeta_0 + \bar{\zeta}_0\right)^2 \tag{1.88}$$

Order (ϵ^2)

$$D_0\zeta_2 - i\omega\zeta_2 = -D_2\zeta_0 - D_1\zeta_1 + \frac{i\delta}{\omega}\left(\zeta_0 + \bar{\zeta}_0\right)\left(\zeta_1 + \bar{\zeta}_1\right) + \frac{i\alpha}{2\omega}\left(\zeta_0 + \bar{\zeta}_0\right)^3 \tag{1.89}$$

The general solution of (1.87) can be expressed as

$$\zeta_0 = A\left(T_1, T_2\right) e^{i\omega T_0} \tag{1.90}$$

where A is an undetermined function of T_1 and T_2 at this order; it is determined by eliminating the secular terms at the next orders of approximation.

Substituting (1.90) into (1.88) yields

$$D_0\zeta_1 - i\omega\zeta_1 = -D_1 A e^{i\omega T_0} + \frac{i\delta}{2\omega}\left(A^2 e^{2i\omega T_0} + 2A\bar{A} + \bar{A}^2 e^{-2i\omega T_0}\right) \tag{1.91}$$

Eliminating the terms that produce secular terms in (1.91) demands that $D_1 A = 0$ or $A = A(T_2)$. Then, the solution of (1.91) can be expressed as

$$\zeta_1 = \frac{\delta A^2}{2\omega^2} e^{2i\omega T_0} - \frac{\delta A\bar{A}}{\omega^2} - \frac{\delta \bar{A}^2}{6\omega^2} e^{-2i\omega T_0} \tag{1.92}$$

where the solution of the homogeneous equation has not been included so that the amplitude of the term at the frequency of oscillation is uniquely defined by the zeroth-order solution (1.90). We note that the coefficients in (1.92) are the same as the Γ_i defined in (1.69).

Substituting (1.90) and (1.92) into (1.89) and using the fact that $D_1 A = 0$, we have

$$D_0\zeta_2 - i\omega\zeta_2 = -D_2 A e^{i\omega T_0} + \frac{i}{2\omega}\left(3\alpha - \frac{10\delta^2}{3\omega^2}\right) A^2\bar{A} e^{i\omega T_0} + \text{NST} \tag{1.93}$$

where NST stands for the terms that do not produce secular terms. Eliminating the terms that produce secular terms from (1.93), we obtain

$$D_2 A = \frac{i}{2\omega}\left(3\alpha - \frac{10\delta^2}{3\omega^2}\right) A^2\bar{A} \tag{1.94}$$

Putting $\xi = Ae^{i\omega t}$ in (1.78) and using the fact that $D_2 A = \epsilon^2 dA/dt$, we obtain exactly (1.94).

We note that, for a uniform second approximation, we do not need to solve for ζ_2, but we only need to inspect (1.93) and eliminate the terms that produce secular terms. Similarly, to determine a uniform second approximation by using the method of normal forms, we do not need to determine h_2 in (1.72), but we need only keep the resonance terms in (1.73).

1.5.3
A Single Transformation

Instead of using successive transformations to produce the normal form of (1.64), one can formulate the process as a perturbation method. Thus, we expand ζ in a power series of ϵ in terms of a new variable η in the form

$$\zeta = \eta + \epsilon h_1(\eta, \bar{\eta}) + \epsilon^2 h_2(\eta, \bar{\eta}) + \cdots \tag{1.95}$$

$$\dot{\eta} = i\omega\eta + \epsilon g_1(\eta, \bar{\eta}) + \epsilon^2 g_2(\eta, \bar{\eta}) + \cdots \tag{1.96}$$

where h_1 and h_2 are smooth functions of η and $\bar{\eta}$ and g_1 and g_2 contain all of the resonance and near-resonance terms. Substituting (1.95) and (1.96) into (1.64) and equating coefficients of like powers of ϵ, we obtain

$$g_1 + i\omega\left(\eta\frac{\partial h_1}{\partial \eta} - \bar{\eta}\frac{\partial h_1}{\partial \bar{\eta}} - h_1\right) = \frac{i\delta}{2\omega}(\eta + \bar{\eta})^2 \tag{1.97}$$

$$g_2 + i\omega\left(\eta\frac{\partial h_2}{\partial \eta} - \bar{\eta}\frac{\partial h_2}{\partial \bar{\eta}} - h_2\right) = -g_1\frac{\partial h_1}{\partial \eta} - \bar{g}_1\frac{\partial h_1}{\partial \bar{\eta}} + \frac{i\alpha}{2\omega}(\eta + \bar{\eta})^3 + \frac{i\delta}{\omega}(\eta + \bar{\eta})\left(h_1 + \bar{h}_1\right) \tag{1.98}$$

Equations 1.97 and 1.98 are the so-called homology equations for h_1 and h_2.

Next, we need to determine g_1 and h_1 from (1.97). In order that h_1 be nonsingular (smooth), we choose g_1 to eliminate all of the resonance and near-resonance terms; otherwise, h_1 will be singular (i.e., have secular terms) if there are resonance terms and near singular (i.e., have small divisors) if there are near-resonance terms. In the present case, there are no resonance terms in (1.97). To see this, we note from (1.96) that $\eta = Be^{i\omega t}$ and hence the perturbation terms on the right-hand side of (1.97) contain terms proportional to $e^{\pm 2i\omega t}$ and a constant. Because none of these terms is proportional to $e^{i\omega t}$, which is the solution of the first-order problem in (1.96), there are no resonance terms. If we are in doubt, we seek a function h_1 that can be used to eliminate all of the perturbation terms. If we are successful in finding a smooth function h_1 that eliminates all of the perturbation terms, then $g_1 = 0$. Otherwise, we choose g_1 to eliminate all terms that produced the troublesome terms (i.e., singular and near-singular terms) in h_1. Because the perturbation terms are of second degree, we let

$$h_1 = \Gamma_1\eta^2 + \Gamma_2\eta\bar{\eta} + \Gamma_3\bar{\eta}^2 \tag{1.99}$$

choose Γ_1, Γ_2, and Γ_3 to eliminate $i\delta(\eta+\bar{\eta})^2/2\omega$ and obtain (1.69). Because the obtained Γ_m are regular, there are no resonance terms and $g_1 = 0$.

Substituting (1.99) and $g_1 = 0$ into (1.98) and using (1.69), we obtain

$$g_2 + i\omega\left(\eta\frac{\partial h_2}{\partial \eta} - \bar{\eta}\frac{\partial h_2}{\partial \bar{\eta}} - h_2\right) = \frac{i\delta^2}{3\omega^3}(\eta+\bar{\eta})\left(\eta^2+\bar{\eta}^2-6\eta\bar{\eta}\right) + \frac{i\alpha}{2\omega}(\eta+\bar{\eta})^3 \tag{1.100}$$

As stated earlier, we do not need to determine h_2 and all that we need is to inspect (1.100) and choose g_2 to eliminate all of the resonance and near-resonance terms. Again, because $\eta \propto e^{i\omega t}$, only the term proportional to $\eta^2\bar{\eta}$ is a resonance term. Hence, choosing g_2 to eliminate this term, we obtain

$$g_2 = \frac{i}{2\omega}\left(3\alpha - \frac{10\delta^2}{3\omega^2}\right)\eta^2\bar{\eta} \tag{1.101}$$

Substituting (1.95) and (1.99) into (1.10) and using (1.69), we obtain

$$u = \eta + \bar{\eta} + \frac{\epsilon\delta}{3\omega^2}\left(\eta^2+\bar{\eta}^2-6\eta\bar{\eta}\right) + \cdots \tag{1.102}$$

Substituting for g_2 from (1.101) into (1.96) and using the fact that $g_1 = 0$, we have

$$\dot{\eta} = i\omega\eta + \frac{i\epsilon^2}{2\omega}\left(3\alpha - \frac{10\delta^2}{3\omega^2}\right)\eta^2\bar{\eta} + \cdots \tag{1.103}$$

in agreement through second order with the expansions obtained in Sections 1.5.1 and 1.5.2.

1.6 A General System with Quadratic and Cubic Nonlinearities

We consider free oscillations of a single-degree-of-freedom system governed by

$$\ddot{u} + \omega^2 u + \epsilon\left(\delta_1 u^2 + \delta_2 \dot{u}^2\right) + \epsilon^2\left(2\mu\dot{u} + \alpha_1 u^3 + \alpha_2 u^2\dot{u} + \alpha_3 u\dot{u}^2 + \alpha_4\dot{u}^3\right) = 0 \tag{1.104}$$

so that

$$f = -\epsilon\left(\delta_1 u^2 + \delta_2\dot{u}^2\right) - \epsilon^2\left(2\mu\dot{u} + \alpha_1 u^3 + \alpha_2 u^2\dot{u} + \alpha_3 u\dot{u}^2 + \alpha_4\dot{u}^3\right) \tag{1.105}$$

We note that a small nondimensional parameter ϵ has been introduced as a bookkeeping device. The quadratic terms have been ordered as $O(\epsilon)$, whereas the cubic terms and the linear damping term have been ordered as $O(\epsilon^2)$. Then, (1.14) becomes

$$\dot{\zeta} = i\omega\zeta + \frac{i\epsilon}{2\omega}\left[\delta_1\left(\zeta+\bar{\zeta}\right)^2 - \delta_2\omega^2\left(\zeta-\bar{\zeta}\right)^2\right] - \epsilon^2\mu\left(\zeta-\bar{\zeta}\right)$$
$$+ \frac{i\epsilon^2}{2\omega}\left[\alpha_1\left(\zeta+\bar{\zeta}\right)^3 + i\omega\alpha_2\left(\zeta+\bar{\zeta}\right)^2\left(\zeta-\bar{\zeta}\right)\right.$$
$$\left.-\omega^2\alpha_3\left(\zeta+\bar{\zeta}\right)\left(\zeta-\bar{\zeta}\right)^2 - i\omega^3\alpha_4\left(\zeta-\bar{\zeta}\right)^3\right] \tag{1.106}$$

As in Section 1.5.3, we seek an expansion for (1.106) in the form (1.95) and (1.96), equate coefficients of like powers of ϵ, and obtain

$$g_1 + i\omega\left(\eta\frac{\partial h_1}{\partial\eta} - \bar{\eta}\frac{\partial h_1}{\partial\bar{\eta}} - h_1\right) = \frac{i}{2\omega}\left[\delta_1(\eta+\bar{\eta})^2 - \delta_2\omega^2(\eta-\bar{\eta})^2\right] \tag{1.107}$$

$$g_2 + i\omega\left(\eta\frac{\partial h_2}{\partial\eta} - \bar{\eta}\frac{\partial h_2}{\partial\bar{\eta}} - h_2\right) = -g_1\frac{\partial h_1}{\partial\eta} - \bar{g}_1\frac{\partial h_1}{\partial\bar{\eta}} - \mu(\eta-\bar{\eta})$$
$$+ \frac{i}{\omega}\left[\delta_1(\eta+\bar{\eta})\left(h_1+\bar{h}_1\right) - \delta_2\omega^2(\eta-\bar{\eta})\left(h_1-\bar{h}_1\right)\right] + \frac{i}{2\omega}\alpha_1(\eta+\bar{\eta})^3$$
$$+ \frac{i}{2\omega}\left[i\omega\alpha_2(\eta+\bar{\eta})^2(\eta-\bar{\eta}) - i\omega^3\alpha_4(\eta-\bar{\eta})^3 - \omega^2\alpha_3(\eta+\bar{\eta})(\eta-\bar{\eta})^2\right] \tag{1.108}$$

The right-hand side of (1.107) does not contain resonance or near-resonance terms, and hence we put $g_1 = 0$, seek h_1 in the form (1.99), and obtain

$$\Gamma_1 = \frac{\delta_1}{2\omega^2} - \frac{1}{2}\delta_2\,, \quad \Gamma_2 = -\frac{\delta_1}{\omega^2} - \delta_2\,, \quad \Gamma_3 = -\frac{\delta_1}{6\omega^2} + \frac{1}{6}\delta_2 \tag{1.109}$$

Because all of the Γ_m are regular, our conclusion that there are no resonance or near-resonance terms in (1.107) and hence $g_1 = 0$ is justified a posteriori.

Substituting (1.99) and (1.109) into (1.108) and using the fact that $g_1 = 0$, we obtain

$$g_2 + i\omega\left(\eta\frac{\partial h_2}{\partial\eta} - \bar{\eta}\frac{\partial h_2}{\partial\bar{\eta}} - h_2\right) = -\frac{2}{3}i\delta_2\omega\left(\frac{\delta_1}{\omega^2} - \delta_2\right)(\eta-\bar{\eta})\left(\eta^2-\bar{\eta}^2\right)$$
$$-\mu(\eta-\bar{\eta}) + \frac{i\delta_1}{3\omega}(\eta+\bar{\eta})\left[\left(\frac{\delta_1}{\omega^2} - \delta_2\right)\left(\eta^2+\bar{\eta}^2\right) - 6\left(\frac{\delta_1}{\omega^2} + \delta_2\right)\eta\bar{\eta}\right]$$
$$+ \frac{i}{2\omega}\left[\alpha_1(\eta+\bar{\eta})^3 + i\omega\alpha_2(\eta+\bar{\eta})^2(\eta-\bar{\eta}) - \omega^2\alpha_3(\eta+\bar{\eta})(\eta-\bar{\eta})^2\right.$$
$$\left.-i\omega^3\alpha_4(\eta-\bar{\eta})^3\right] + \cdots \tag{1.110}$$

Inspecting the right-hand side of (1.110), we conclude that the terms proportional to η and $\eta^2\bar{\eta}$ are the only resonance terms and there are no near-resonance terms. Consequently, choosing g_2 to eliminate the resonance terms, we obtain

$$g_2 = -\mu\eta + \frac{i}{2\omega}\left[3\alpha_1 + \omega^2\alpha_3 - \frac{2}{3}\left(\frac{5\delta_1^2}{\omega^2} + 5\delta_1\delta_2 + 2\delta_2^2\omega^2\right)\right.$$
$$\left.+i\omega\left(\alpha_2 + 3\omega^2\alpha_4\right)\right]\eta^2\bar{\eta} \tag{1.111}$$

Substituting (1.95) and (1.99) into (1.10) and using (1.109), we obtain

$$u = \eta + \bar{\eta} + \epsilon \left[\left(\frac{\delta_1}{3\omega^2} - \frac{1}{3}\delta_2 \right) (\eta^2 + \bar{\eta}^2) - \left(\frac{2\delta_1}{\omega^2} + 2\delta_2 \right) \eta\bar{\eta} \right] + \cdots \quad (1.112)$$

Substituting for g_2 from (1.111) into (1.96) and using the fact that $g_1 = 0$, we obtain

$$\dot{\eta} = i\omega\eta - \epsilon^2\mu\eta + \frac{i\epsilon^2}{2\omega} \left[3\alpha_1 + \omega^2\alpha_3 - \frac{2}{3} \left(\frac{5\delta_1^2}{\omega^2} + 5\delta_1\delta_2 + 2\delta_2^2\omega^2 \right) \right.$$
$$\left. + i\omega \left(\alpha_2 + 3\omega^2\alpha_4 \right) \right] \eta^2\bar{\eta} \quad (1.113)$$

To compare the expansion (1.112) and (1.113) with that obtained by using the method of multiple scales (Nayfeh, 1984), we substitute the polar form

$$\eta = \tfrac{1}{2} a e^{i(\omega t + \beta)} \quad (1.114)$$

in (1.112) and (1.113) and obtain

$$u = a\cos(\omega t + \beta) + \frac{1}{6}\epsilon a^2 \left[\left(\frac{\delta_1}{\omega^2} - \delta_2 \right) \cos(2\omega t + 2\beta) - 3\left(\frac{\delta_1}{\omega^2} + \delta_2 \right) \right]$$
$$+ \cdots \quad (1.115)$$

where

$$\dot{a} = -\epsilon^2\mu a - \tfrac{1}{8}\epsilon^2 \left(\alpha_2 + 3\omega^2\alpha_4 \right) a^3 + \cdots \quad (1.116)$$

$$a\dot{\beta} = \frac{\epsilon^2}{8\omega} \left[3\alpha_1 + \omega^2\alpha_3 - \frac{2}{3} \left(\frac{5\delta_1}{\omega^2} + 5\delta_1\delta_2 + 2\delta_2^2\omega^2 \right) \right] a^3 + \cdots \quad (1.117)$$

which is formally equivalent to that obtained by using the method of multiple scales.

1.7
The van der Pol Oscillator

In this section, we construct a second-order approximation of the normal form of the van der Pol oscillator

$$\ddot{u} + \omega^2 u = \epsilon(1 - u^2)\dot{u} \quad (1.118)$$

Using the transformation (1.10), we rewrite (1.118) as

$$\dot{\zeta} = i\omega\zeta + \tfrac{1}{2}\epsilon(\zeta - \bar{\zeta}) \left[1 - (\zeta + \bar{\zeta})^2 \right] \quad (1.119)$$

1.7.1 The Method of Normal Forms

As in the preceding two sections, we seek a second-order expansion of (1.119) in the form (1.95) and (1.96), equate coefficients of like powers of ϵ, and obtain

$$g_1 + i\omega\left(\eta\frac{\partial h_1}{\partial\eta} - \bar{\eta}\frac{\partial h_1}{\partial\bar{\eta}} - h_1\right) = \frac{1}{2}(\eta - \bar{\eta}) - \frac{1}{2}\left(\eta^3 + \eta^2\bar{\eta} - \bar{\eta}^2\eta - \bar{\eta}^3\right) \tag{1.120}$$

$$g_2 + i\omega\left(\eta\frac{\partial h_2}{\partial\eta} - \bar{\eta}\frac{\partial h_2}{\partial\bar{\eta}} - h_2\right) = -g_1\frac{\partial h_1}{\partial\eta} - \bar{g}_1\frac{\partial h_1}{\partial\bar{\eta}} + \frac{1}{2}\left(h_1 - \bar{h}_1\right)$$
$$-\frac{3}{2}\eta^2 h_1 - \eta\bar{\eta}(h_1 - \bar{h}_1) - \frac{1}{2}\eta^2\bar{h}_1 + \frac{1}{2}\bar{\eta}^2 h_1 + \frac{3}{2}\bar{\eta}^2\bar{h}_1 \tag{1.121}$$

Choosing g_1 to eliminate the resonance terms in (1.120), we have

$$g_1 = \tfrac{1}{2}\eta - \tfrac{1}{2}\eta^2\bar{\eta} \tag{1.122}$$

Then, we seek h_1 in the form

$$h_1 = \Delta_1\bar{\eta} + \Lambda_1\eta^3 + \Lambda_2\eta\bar{\eta}^2 + \Lambda_3\bar{\eta}^3 \tag{1.123}$$

Substituting (1.123) and (1.122) into (1.120) yields

$$\left(2i\omega\Delta_1 - \tfrac{1}{2}\right)\bar{\eta} - \left(2i\omega\Lambda_1 + \tfrac{1}{2}\right)\eta^3 + \left(2i\omega\Lambda_2 + \tfrac{1}{2}\right)\eta\bar{\eta}^2$$
$$+ \left(4i\omega\Lambda_3 + \tfrac{1}{2}\right)\bar{\eta}^3 = 0 \tag{1.124}$$

Hence,

$$\Delta_1 = -\frac{i}{4\omega}, \quad \Lambda_1 = \frac{i}{4\omega}, \quad \Lambda_2 = \frac{i}{4\omega}, \quad \Lambda_3 = \frac{i}{8\omega} \tag{1.125}$$

Therefore,

$$h_1 = -\frac{1}{4\omega}i\left(\bar{\eta} - \eta^3 - \eta\bar{\eta}^2 - \frac{1}{2}\bar{\eta}^3\right) \tag{1.126}$$

Substituting (1.122) and (1.126) into (1.121) yields

$$g_2 + i\omega\left(\eta\frac{\partial h_2}{\partial\eta} - \bar{\eta}\frac{\partial h_2}{\partial\bar{\eta}} - h_2\right) = -\frac{i}{16\omega}\left(2\eta + 5\eta^3 + 5\eta^5 - 12\eta^2\bar{\eta}\right.$$
$$\left.-2\eta^4\bar{\eta} - 4\eta\bar{\eta}^2 + 11\eta^3\bar{\eta}^2 + 2\bar{\eta}^3 + 5\eta^2\bar{\eta}^3 + \eta\bar{\eta}^4 + 5\bar{\eta}^5\right) \tag{1.127}$$

Choosing g_2 to eliminate the resonance terms (terms proportional to η, $\eta^2\bar{\eta}$, and $\eta^3\bar{\eta}^2$) from (1.127), we have

$$g_2 = -\frac{1}{16\omega}i\left(2\eta - 12\eta^2\bar{\eta} + 11\eta^3\bar{\eta}^2\right) \tag{1.128}$$

Substituting (1.122) and (1.128) into (1.96), we obtain, to the second approximation, the normal form

$$\dot{\eta} = i\omega\eta + \frac{1}{2}\epsilon\left(\eta - \eta^2\bar{\eta}\right) - \frac{1}{16\omega}i\epsilon^2\left(2\eta - 12\eta^2\bar{\eta} + 11\eta^3\bar{\eta}^2\right) \tag{1.129}$$

Substituting the polar form (1.27) into (1.129) and separating real and imaginary parts, we obtain

$$\dot{a} = \tfrac{1}{2}\epsilon\left(a - \tfrac{1}{4}a^3\right) \tag{1.130}$$

$$\dot{\beta} = -\frac{1}{8\omega}\epsilon^2\left(1 - \frac{3}{2}a^2 + \frac{11}{32}a^4\right) \tag{1.131}$$

in agreement with those obtained with the generalized method of averaging (Nayfeh, 1973).

1.7.2
The Method of Multiple Scales

We seek a second-order expansion of the solution of (1.119) in the form (1.85). Substituting (1.85) and (1.86) into (1.119) and equating coefficients of like powers of ϵ, we obtain

Order (ϵ^0)

$$D_0\zeta_0 - i\omega\zeta_0 = 0 \tag{1.132}$$

Order (ϵ)

$$D_0\zeta_1 - i\omega\zeta_1 = -D_1\zeta_0 + \tfrac{1}{2}\left(\zeta_0 - \bar{\zeta}_0\right) - \tfrac{1}{2}\left(\zeta_0^3 + \zeta_0^2\bar{\zeta}_0 - \zeta_0\bar{\zeta}_0^2 - \bar{\zeta}_0^3\right) \tag{1.133}$$

Order (ϵ^2)

$$\begin{aligned} D_0\zeta_2 - i\omega\zeta_2 = &-D_2\zeta_0 - D_1\zeta_1 + \tfrac{1}{2}\left(\zeta_1 - \bar{\zeta}_1\right) - \tfrac{1}{2}\left(3\zeta_0^2 + 2\zeta_0\bar{\zeta}_0 - \bar{\zeta}_0^2\right)\zeta_1 \\ &- \tfrac{1}{2}\left(\zeta_0^2 - 2\zeta_0\bar{\zeta}_0 - 3\bar{\zeta}_0^2\right)\bar{\zeta}_1 \end{aligned} \tag{1.134}$$

The general solution of (1.132) can be expressed as in (1.90). Then, (1.133) becomes

$$\begin{aligned} D_0\zeta_1 - i\omega\zeta_1 = &-D_1Ae^{i\omega T_0} + \tfrac{1}{2}Ae^{i\omega T_0} - \tfrac{1}{2}\bar{A}e^{-i\omega T_0} - \tfrac{1}{2}A^3e^{3i\omega T_0} \\ &- \tfrac{1}{2}A^2\bar{A}e^{i\omega T_0} + \tfrac{1}{2}A\bar{A}^2e^{-i\omega T_0} + \tfrac{1}{2}\bar{A}^3e^{-3i\omega T_0} \end{aligned} \tag{1.135}$$

Eliminating the terms that lead to secular terms from (1.135) yields

$$D_1A = \tfrac{1}{2}\left(A - A^2\bar{A}\right) \tag{1.136}$$

Then, the solution of (1.135) can be expressed as

$$\zeta_1 = \frac{i}{8\omega}\left(-2\bar{A}e^{-i\omega T_0} + 2A^3e^{3i\omega T_0} + 2A\bar{A}^2e^{-i\omega T_0} + \bar{A}^3e^{-3i\omega T_0}\right) \tag{1.137}$$

Substituting (1.90), (1.136), and (1.137) into (1.134), we have

$$D_0\zeta_2 - i\omega\zeta_2 = -D_2Ae^{i\omega T_0} - \frac{i}{16\omega}\left[2A - 12A^2\bar{A} + 11A^3\bar{A}^2\right]e^{i\omega T_0} + \text{NST} \tag{1.138}$$

Eliminating the terms that lead to secular terms from (1.138) yields

$$D_2A = -\frac{i}{16\omega}\left[2A - 12A^2\bar{A} + 11A^3\bar{A}^2\right] \tag{1.139}$$

Using the method of reconstitution, we obtain from (1.136) and (1.139) that

$$\dot{A} = \frac{1}{2}\epsilon\left(A - A^2\bar{A}\right) - \frac{i}{16\omega}\epsilon^2\left[2A - 12A^2\bar{A} + 11A^3\bar{A}^2\right] \tag{1.140}$$

Letting $\zeta = Ae^{i\omega t}$ in (1.129), we obtain (1.140), which means that the results obtained with the methods of normal forms and multiple scales are the same.

1.8 Exercises

1.8.1 Use the methods of normal forms and multiple scales to determine the normal forms of

a) $\ddot{u} + \omega^2 u + \alpha\dot{u}^3 = 0$,
b) $\ddot{u} + \omega^2 u + \alpha u^2\dot{u} = 0$,
c) $\ddot{u} + \omega^2 u + \alpha u^5 = 0$,
d) $\ddot{u} + \omega^2 u + \alpha u^3\dot{u}^2 = 0$,
e) $\ddot{u} + \omega^2 u + \epsilon\dot{u}^5 = 0$.

1.8.2 Use the methods of multiple scales and normal forms to construct a second-order approximation to the normal form of

$$\ddot{u} + \omega^2 u + \alpha u^3 + 2\mu u^2\dot{u} = 0$$

1.8.3 Use the methods of multiple scales and normal forms to construct a first-order approximation to the normal form of

$$\ddot{u} + \omega^2 u + \alpha u^2\ddot{u} = 0$$

1.8.4 Use the methods of normal forms and multiple scales to construct a second-order approximation to the normal form of

$$\ddot{u} + \omega^2 u + \alpha\left(\dot{u} - \tfrac{1}{3}\dot{u}^3\right) = 0$$

1.8.5 Consider the equation

$$\ddot{x} + x + 3x^2 + 2x^3 = 0$$

Determine the equilibrium points. Determine, to second order, the normal form of the system near each of these equilibrium points.

1.8.6 Consider

$$\ddot{x} + x + ax^2 + 2x^3 = 0$$

Show that there is only one equilibrium point when $a < 2\sqrt{2}$ and that there are three equilibrium points when $a > 2\sqrt{2}$. Determine, to second order, the normal forms near the equilibrium points.

1.8.7 Consider

$$\ddot{x} + x - \frac{a}{1-x} = 0$$

Show that there is only one equilibrium point when $a \leq 1/4$. Determine, to second order, the normal form near this equilibrium point.

1.8.8 Consider

$$\ddot{x} - 3x + x^3 = -2$$

Determine the equilibrium points and the normal forms near them.

1.8.9 Consider

$$\ddot{u} - u + u^4 = 0$$

Determine the equilibrium points and the normal forms near them.

1.8.10 Consider

$$\ddot{x} - 2x - x^2 + x^3 = 0$$

Determine the equilibrium points and the normal forms near them.

1.8.11 Consider

$$\ddot{u} + u - \frac{3}{16(1-u)} = 0$$

Determine the equilibrium points and the normal forms near them.

1.8.12 Use the methods of multiple scales and normal forms to determine a first-order uniform expansion for the general solution of

$$\ddot{\theta} + \omega^2 \sin\theta + \frac{4\sin^2\theta}{1 + 4(1-\cos\theta)}\dot{\theta} = 0$$

for small but finite θ.

1.8.13 Consider the equation

$$\ddot{u} + \omega_0^2 u + \frac{\mu \dot{u}}{1 - u^2} = 0$$

Use the methods of multiple scales and normal forms to determine a first-order uniform expansion for small u.

1.8.14 Consider the equation

$$\left(l^2 + r^2 - 2rl \cos\theta\right) \ddot{\theta} + rl \sin\theta\, \dot{\theta}^2 + gl \sin\theta = 0$$

where g, r, and l are constants. Determine a first-order expansion for small but finite θ by using the methods of multiple scales and normal forms.

1.8.15 Consider the equation

$$\left(\tfrac{1}{12} l^2 + r^2\theta^2\right) \ddot{\theta} + r^2\theta\, \dot{\theta}^2 + gr\theta \cos\theta = 0$$

where r, l, and g are constants. Determine a first-order uniform expansion for small but finite θ by using the methods of multiple scales and normal forms.

1.8.16 The motion of a simple pendulum is governed by

$$\ddot{\theta} + \frac{g}{l} \sin\theta = 0$$

Use the methods of multiple scales and normal forms to determine a first-order uniform expansion for small but finite θ.

1.8.17 Consider the equation

$$\ddot{\theta} = \Omega^2 \sin\theta \cos\theta - \frac{g}{R} \sin\theta$$

Use the methods of multiple scales and normal forms to determine a first-order uniform expansion for small but finite θ.

1.8.18 The motion of a particle on a rotating parabola is governed by

$$\left(1 + 4p^2x^2\right) \ddot{x} + \Lambda x + 4p^2\dot{x}^2 = 0$$

where p and Λ are constants. Use the methods of multiple scales and normal forms to determine a first-order expansion for small but finite x.

1.8.19 Consider the equation

$$\left(1 + \frac{u^2}{1 - u^2}\right) \ddot{u} + \frac{u\dot{u}^2}{(1 - u^2)^2} + \omega_0^2 u + \frac{g}{l} \frac{u}{\sqrt{1 - u^2}} = 0$$

Use the methods of multiple scales and normal forms to determine a first-order expansion for small u.

2
Systems of First-Order Equations

2.1
Introduction

In this chapter, we describe in detail the problem of calculating the normal form for a system of first-order equations having the form

$$\dot{\boldsymbol{u}} = A\boldsymbol{u} + \epsilon \boldsymbol{F}_1(\boldsymbol{u}) + \epsilon^2 \boldsymbol{F}_2(\boldsymbol{u}) + \cdots \tag{2.1}$$

where $\boldsymbol{u}$ and the $\boldsymbol{F}_m$ are column vectors of length n, A is an $n \times n$ constant matrix, and ϵ is a small nondimensional parameter, which may represent a physical quantity, or it may be just a bookkeeping device and set equal to unity in the final result. In this book, the normalization is carried out in terms of ϵ, whereas in the literature the normalization is usually carried out in terms of the degree of the polynomials in the nonlinear terms. In the latter case, the components of $\boldsymbol{F}_m(\boldsymbol{u})$ are homogeneous polynomials of degree $m + 1$ in $\boldsymbol{u}$. We assume that the $\boldsymbol{F}_m$ are smooth vector fields satisfying $\boldsymbol{F}_m(0) = 0$ so that $\boldsymbol{u} = 0$ is a fixed point of (2.1).

As a first step, we introduce a linear transformation $\boldsymbol{u} = P\boldsymbol{x}$, where P is a nonsingular or invertible matrix, in (2.1) and obtain

$$P\dot{\boldsymbol{x}} = AP\boldsymbol{x} + \epsilon \boldsymbol{F}_1(P\boldsymbol{x}) + \epsilon^2 \boldsymbol{F}_2(P\boldsymbol{x}) + \cdots \tag{2.2}$$

Multiplying (2.2) from the left by P^{-1}, the inverse of P, we obtain

$$\dot{\boldsymbol{x}} = J\boldsymbol{x} + \epsilon \boldsymbol{f}_1(\boldsymbol{x}) + \epsilon^2 \boldsymbol{f}_2(\boldsymbol{x}) + \cdots \tag{2.3}$$

where

$$J = P^{-1}AP \quad \text{and} \quad \boldsymbol{f}_m(\boldsymbol{x}) = P^{-1}\boldsymbol{F}_m(P\boldsymbol{x}) \tag{2.4}$$

We choose P so that J has a simple real form.

The simplest possible real form for J depends on the eigenvalues λ_i and eigenvectors $\boldsymbol{p}_i$ of A. There are three possibilities (Coddington and Levinson, 1955; Hale, 1980; Hirsch and Smale, 1974; Walter, 1976):

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a) All eigenvalues are real and distinct.
b) Some eigenvalues occur in complex conjugate pairs.
c) Some eigenvalues are repeated.

In the first case, there are n linearly independent real eigenvectors $\boldsymbol{p}_i$ of A. Then, forming the matrix P by having its columns be the eigenvectors $\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n$ produces a diagonal matrix J with real entries $\lambda_1, \lambda_2, \ldots, \lambda_n$. For two-dimensional systems,

$$J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

When the eigenvalues are real and some of them are repeated, there are two possibilities. First, there are n linearly independent real eigenvectors $\boldsymbol{p}_i$. Then, choosing P as in the first case produces a diagonal matrix with real entries $\lambda_1, \lambda_2, \ldots, \lambda_n$, where some of them are repeated. For two-dimensional systems,

$$J = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

Second, there are less than n linearly independent eigenvectors. In this case, we form the matrix P by having its columns $\boldsymbol{p}_i$ be the generalized eigenvectors of A; that is, the eigenvectors of $(A - \lambda_j I)^m$, where m is the multiplicity of the eigenvalue λ_j. With this choice for P, J takes the so-called Jordan form. For two-dimensional systems,

$$J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad \text{or} \quad J = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$$

When some of the eigenvalues occur in complex conjugate pairs, some of the eigenvectors are complex. Forming the matrix P by having its columns be the generalized real eigenvectors of A produces a matrix J that is in the so-called real Jordan form. For two-dimensional systems,

$$J = \begin{bmatrix} \alpha & -\omega \\ \omega & \alpha \end{bmatrix} \quad \text{or} \quad J = \begin{bmatrix} \alpha & \omega \\ -\omega & \alpha \end{bmatrix}$$

where α and ω are real constants.

The fixed point $\boldsymbol{u} = 0$ is called hyperbolic if none of the eigenvalues of A has a zero real part and nonhyperbolic if at least one of the eigenvalues of A has a zero real part. The zero real parts imply the existence of constraints on the elements of A. For two-dimensional systems, there are four possible constraints:

1. A has real eigenvalues and one of them is zero; that is,

$$|A| = 0 \quad \text{and} \quad \text{Tr}(A) \neq 0$$

where $\text{Tr}(A)$ is the trace of A.

2. A has purely imaginary eigenvalues; that is,

$$|A| \neq 0 \quad \text{and} \quad \mathrm{Tr}(A) = 0$$

3. Both eigenvalues of A are zero but A is not the null matrix; that is,

$$A \neq 0\,, \quad |A| = 0\,, \quad \text{and} \quad \mathrm{Tr}(A) = 0$$

4. $A = 0$.

The equality constraints are called degeneracy conditions. The number of degeneracy conditions indicates the level of degeneracy or codimension of the fixed point (Dumortier, 1977; Guckenheimer and Holmes, 1983).

Our aim is to construct a sequence of transformations that successively remove the perturbation terms $\boldsymbol{f}_m$, starting from $\boldsymbol{f}_1$. Ideally, we would like to be able to remove all of the $\boldsymbol{f}_m$, especially, if they are nonlinear, thereby reducing (2.3) to a linear problem. In general, this is not possible, as discussed below.

Instead of using a sequence of transformations, we introduce the single near-identity transformation

$$\boldsymbol{x} = \boldsymbol{y} + \epsilon \boldsymbol{h}_1(\boldsymbol{y}) + \epsilon^2 \boldsymbol{h}_2(\boldsymbol{y}) + \cdots \tag{2.5}$$

into (2.3) and choose the $\boldsymbol{h}_m$ so that it takes the simplest possible form, the so-called normal form

$$\dot{\boldsymbol{y}} = J\boldsymbol{y} + \epsilon \boldsymbol{g}_1(\boldsymbol{y}) + \epsilon^2 \boldsymbol{g}_2(\boldsymbol{y}) + \cdots \tag{2.6}$$

As discussed later, the $\boldsymbol{g}_n$ are referred to as resonance and near-resonance terms. Substituting (2.5) into (2.3) yields

$$\begin{aligned} &\dot{\boldsymbol{y}} + \epsilon D\boldsymbol{h}_1(\boldsymbol{y})\dot{\boldsymbol{y}} + \epsilon^2 D\boldsymbol{h}_2(\boldsymbol{y})\dot{\boldsymbol{y}} + \cdots \\ &= J\boldsymbol{y} + \epsilon J\boldsymbol{h}_1(\boldsymbol{y}) + \epsilon^2 J\boldsymbol{h}_2(\boldsymbol{y}) + \cdots + \epsilon \boldsymbol{f}_1\left[\boldsymbol{y} + \epsilon \boldsymbol{h}_1(\boldsymbol{y}) + \epsilon^2 \boldsymbol{h}_2(\boldsymbol{y}) + \cdots\right] \\ &\quad + \epsilon^2 \boldsymbol{f}_2\left[\boldsymbol{y} + \epsilon \boldsymbol{h}_1(\boldsymbol{y}) + \epsilon^2 \boldsymbol{h}_2(\boldsymbol{y}) + \cdots\right] + \cdots \end{aligned} \tag{2.7}$$

where $D\boldsymbol{h}$ is the Jacobian of $\boldsymbol{h}$. Using (2.6) to eliminate $\dot{\boldsymbol{y}}$ from (2.7) and equating coefficients of like powers of ϵ, we obtain

$$\boldsymbol{g}_1(\boldsymbol{y}) + D\boldsymbol{h}_1(\boldsymbol{y})J\boldsymbol{y} - J\boldsymbol{h}_1(\boldsymbol{y}) = \boldsymbol{f}_1(\boldsymbol{y}) \tag{2.8}$$

$$\boldsymbol{g}_2(\boldsymbol{y}) + D\boldsymbol{h}_2(\boldsymbol{y})J\boldsymbol{y} - J\boldsymbol{h}_2(\boldsymbol{y}) = \boldsymbol{f}_2(\boldsymbol{y}) + D\boldsymbol{f}_1(\boldsymbol{y})\boldsymbol{h}_1(\boldsymbol{y}) - D\boldsymbol{h}_1(\boldsymbol{y})\boldsymbol{g}_1(\boldsymbol{y}) \tag{2.9}$$

The operator $\mathcal{L}(\boldsymbol{h}_1(\boldsymbol{y})) = D\boldsymbol{h}_1(\boldsymbol{y})J\boldsymbol{y} - J\boldsymbol{h}_1(\boldsymbol{y}) = [\boldsymbol{h}_1, J\boldsymbol{y}]$ is called the *Lie* or *Poisson bracket*. We note that although the $\boldsymbol{h}_n(\boldsymbol{y})$, which describe the transformation (2.5), are in general nonlinear functions of $\boldsymbol{y}$, they are found by solving a sequence of linear problems, as described below.

Next, we carry out the details of determining the $\boldsymbol{h}_m$ and $\boldsymbol{g}_m$ for specific problems.

2.2 A Two-Dimensional System with Diagonal Linear Part

As a first example, we consider the two-dimensional system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \boldsymbol{x} + \epsilon \begin{bmatrix} \alpha_1 x_1^2 + \alpha_2 x_1 x_2 + \alpha_3 x_2^2 \\ \alpha_4 x_1^2 + \alpha_5 x_1 x_2 + \alpha_6 x_2^2 \end{bmatrix} \tag{2.10}$$

where the λ_i and α_i may be real or complex, but both of λ_1 and λ_2 are not zero.

Using (2.10), we rewrite (2.8) as

$$\begin{bmatrix} g_{11} \\ g_{12} \end{bmatrix} + \begin{bmatrix} \dfrac{\partial h_{11}}{\partial y_1} & \dfrac{\partial h_{11}}{\partial y_2} \\ \dfrac{\partial h_{12}}{\partial y_1} & \dfrac{\partial h_{12}}{\partial y_2} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \end{bmatrix}$$
$$= \begin{bmatrix} \alpha_1 y_1^2 + \alpha_2 y_1 y_2 + \alpha_3 y_2^2 \\ \alpha_4 y_1^2 + \alpha_5 y_1 y_2 + \alpha_6 y_2^2 \end{bmatrix} \tag{2.11}$$

where (h_{11}, h_{12}) and (g_{11}, g_{12}) are the components of $\boldsymbol{h}_1$ and $\boldsymbol{g}_1$. The right-hand side of (2.11) suggests seeking the g_{1m} and h_{1m} in the form

$$h_{11} = \Gamma_1 y_1^2 + \Gamma_2 y_1 y_2 + \Gamma_3 y_2^2 \tag{2.12}$$

$$h_{12} = \Gamma_4 y_1^2 + \Gamma_5 y_1 y_2 + \Gamma_6 y_2^2 \tag{2.13}$$

$$g_{11} = \Lambda_1 y_1^2 + \Lambda_2 y_1 y_2 + \Lambda_3 y_2^2 \tag{2.14}$$

$$g_{12} = \Lambda_4 y_1^2 + \Lambda_5 y_1 y_2 + \Lambda_6 y_2^2 \tag{2.15}$$

Substituting (2.12)–(2.15) into (2.11) yields

$$\lambda_1 y_1 (2\Gamma_1 y_1 + \Gamma_2 y_2) + \lambda_2 y_2 (\Gamma_2 y_1 + 2\Gamma_3 y_2) - \lambda_1 \left(\Gamma_1 y_1^2 + \Gamma_2 y_1 y_2 + \Gamma_3 y_2^2\right)$$
$$= (\alpha_1 - \Lambda_1)\, y_1^2 + (\alpha_2 - \Lambda_2)\, y_1 y_2 + (\alpha_3 - \Lambda_3)\, y_2^2 \tag{2.16}$$

$$\lambda_1 y_1 (2\Gamma_4 y_1 + \Gamma_5 y_2) + \lambda_2 y_2 (\Gamma_5 y_1 + 2\Gamma_6 y_2) - \lambda_2 \left(\Gamma_4 y_1^2 + \Gamma_5 y_1 y_2 + \Gamma_6 y_2^2\right)$$
$$= (\alpha_4 - \Lambda_4)\, y_1^2 + (\alpha_5 - \Lambda_5)\, y_1 y_2 + (\alpha_6 - \Lambda_6)\, y_2^2 \tag{2.17}$$

Equating the coefficients of y_1^2, $y_1 y_2$, and y_2^2 on both sides of (2.16) and (2.17) yields

$$\boldsymbol{B\Gamma} = \boldsymbol{\alpha} - \boldsymbol{\Lambda} \tag{2.18}$$

where $\boldsymbol{\Gamma}$, $\boldsymbol{\alpha}$, and $\boldsymbol{\Lambda}$ are column vectors having the components Γ_m, α_m, and Λ_m, respectively, and

$$B = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\lambda_2 - \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\lambda_1 - \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} \tag{2.19}$$

Thus, the matrix B is diagonal having the eigenvalues $\lambda_1, \lambda_2, 2\lambda_2 - \lambda_1$, and $2\lambda_1 - \lambda_2$. As shown in Section 2.4, these eigenvalues are related to the eigenvalues λ_1 and λ_2 of J and hence A by

$$\lambda_{\boldsymbol{m},i} = m_1\lambda_1 + m_2\lambda_2 - \lambda_i \tag{2.20}$$

where $\boldsymbol{m} = (m_1, m_2)$, m_1 and m_2 are positive integers such that $m_1 + m_2 = 2$ and $m_1, m_2 = 0, 1$, and 2.

Clearly, the matrix B has an inverse and hence (2.18) has a solution for any $\boldsymbol{\alpha}$ if none of its eigenvalues vanishes. Consequently, we put $\boldsymbol{\Lambda} = 0$, explicitly solve (2.18) for the Γ_m, and obtain

$$\Gamma_1 = \frac{\alpha_1}{\lambda_1}, \quad \Gamma_2 = \frac{\alpha_2}{\lambda_2}, \quad \Gamma_3 = \frac{\alpha_3}{2\lambda_2 - \lambda_1},$$
$$\Gamma_4 = \frac{\alpha_4}{2\lambda_1 - \lambda_2}, \quad \Gamma_5 = \frac{\alpha_5}{\lambda_1}, \quad \Gamma_6 = \frac{\alpha_6}{\lambda_2} \tag{2.21}$$

With this choice, all of the nonlinear terms are eliminated from (2.10), resulting in the linear normal form $\dot{\boldsymbol{y}} = J\boldsymbol{y}$.

When any of the eigenvalues $\lambda_1, \lambda_2, 2\lambda_2 - \lambda_1$, and $2\lambda_1 - \lambda_2$ vanishes (i.e., $m_1\lambda_1 + m_2\lambda_2 = \lambda_i$, for some choice of m_1 and m_2), the matrix B is singular and one or more of the Γ_m are unbounded, implying that one cannot eliminate all of the nonlinear terms in (2.10) and hence obtain a linear normal form. The terms that cannot be eliminated are called *resonance terms*. Moreover, the condition $m_1\lambda_1 + m_2\lambda_2 = \lambda_i$ is called a *resonance condition* of order 2 because the nonlinearity is quadratic (i.e., $m_1 + m_2 = 2$). When any of the eigenvalues of B is small, it follows from (2.21) that at least one of the Γ_m has a *small divisor* and therefore the transformation breaks down. The terms that produce small divisors are called *near-resonance terms*. For example, let us consider the resonance condition $2\lambda_2 - \lambda_1 \approx 0$. In this case, it follows from (2.21) that all Γ_m except Γ_3 are regular and that Γ_3 is either singular or has a small divisor. Thus, we choose $\Lambda_3 = \alpha_3$ to avoid this singularity, put the remaining Λ_m equal to zero, and choose $\Gamma_1, \Gamma_2, \Gamma_4, \Gamma_5$, and Γ_6 as in (2.21). Consequently, the normal form of (2.10) is

$$\dot{\boldsymbol{y}} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \boldsymbol{y} + \epsilon \begin{bmatrix} \alpha_3 y_2^2 \\ 0 \end{bmatrix} \tag{2.22}$$

When $2\lambda_1 - \lambda_2 \approx 0$, we choose $\Lambda_4 = \alpha_4$ to avoid the singularity, set the remaining Λ_m equal to zero, and choose $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_5$, and Γ_6 as in (2.21). In this case, the normal form of (2.10) is

$$\dot{y} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} y + \epsilon \begin{bmatrix} 0 \\ \alpha_4 y_1^2 \end{bmatrix} \tag{2.23}$$

When $\lambda_1 \approx 0$, we choose $\Lambda_1 = \alpha_1$ and $\Lambda_5 = \alpha_5$ to avoid the singularities, set the remaining Λ_m equal to zero, and choose $\Gamma_2, \Gamma_3, \Gamma_4$, and Γ_6 as in (2.21). The resulting normal form of (2.10) is

$$\dot{y} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} y + \epsilon \begin{bmatrix} \alpha_1 y_1^2 \\ \alpha_5 y_1 y_2 \end{bmatrix} \tag{2.24}$$

When $\lambda_2 \approx 0$, we choose $\Lambda_2 = \alpha_2$ and $\Lambda_6 = \alpha_6$, set the remaining Λ_m equal to zero, and choose $\Gamma_1, \Gamma_3, \Gamma_4$, and Γ_5 as in (2.21). Consequently, the normal form of (2.10) becomes

$$\dot{y} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} y + \epsilon \begin{bmatrix} \alpha_2 y_1 y_2 \\ \alpha_6 y_2^2 \end{bmatrix} \tag{2.25}$$

Recapping, we note that the 6×6 matrix B maps six-dimensional vectors in the space $\Re^6$ into six-dimensional vectors in $\Re^6$; that is, for any vector Γ in $\Re^6$, $B\Gamma$ is a vector in $\Re^6$. It is clear that the $\boldsymbol{e}_i$, where $\boldsymbol{e}_i$ is the ith member of the natural basis of $\Re^6$ ($\boldsymbol{e}_i$ is a six-dimensional vector whose elements are all zero except the ith element is 1), form a basis for $\Re^6$, and that they are the eigenvectors of B corresponding to the eigenvalues $\lambda_{m,i}$. Then, the subset of eigenvectors of B with nonzero eigenvalues form a basis for the image X of $\Re^6$ under B. Consequently, the components of α lying in X can be eliminated by a proper choice of Γ. Then, the component X^c of $\Re^6$, which does not lie in X and represents a subspace complementary to X, is spanned by the eigenvectors of B corresponding to zero or near-zero eigenvalues. Consequently, the resonance and near-resonance terms are in the subspace of $\Re^6$ spanned by the eigenvectors of B corresponding to zero or near-zero eigenvalues.

Next, we consider system (2.10) with cubic, rather than quadratic, nonlinearities; that is,

$$\dot{x} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} x + \epsilon \begin{bmatrix} \alpha_1 x_1^3 + \alpha_2 x_1^2 x_2 + \alpha_3 x_1 x_2^2 + \alpha_4 x_2^3 \\ \alpha_5 x_1^3 + \alpha_6 x_1^2 x_2 + \alpha_7 x_1 x_2^2 + \alpha_8 x_2^3 \end{bmatrix} \tag{2.26}$$

Then, (2.8) becomes

$$\begin{bmatrix} g_{11} \\ g_{12} \end{bmatrix} + \begin{bmatrix} \dfrac{\partial h_{11}}{\partial y_1} & \dfrac{\partial h_{11}}{\partial y_2} \\ \dfrac{\partial h_{12}}{\partial y_1} & \dfrac{\partial h_{12}}{\partial y_2} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \end{bmatrix}$$
$$= \begin{bmatrix} \alpha_1 y_1^3 + \alpha_2 y_1^2 y_2 + \alpha_3 y_1 y_2^2 + \alpha_4 y_2^3 \\ \alpha_5 y_1^3 + \alpha_6 y_1^2 y_2 + \alpha_7 y_1 y_2^2 + \alpha_8 y_2^3 \end{bmatrix} \tag{2.27}$$

The form of the terms on the right-hand side of (2.27) suggests seeking h_{11}, h_{12}, g_{11}, and g_{12} in the form

$$h_{11} = \Gamma_1 y_1^3 + \Gamma_2 y_1^2 y_2 + \Gamma_3 y_1 y_2^2 + \Gamma_4 y_2^3 \tag{2.28}$$

$$h_{12} = \Gamma_5 y_1^3 + \Gamma_6 y_1^2 y_2 + \Gamma_7 y_1 y_2^2 + \Gamma_8 y_2^3 \tag{2.29}$$

$$g_{11} = \Lambda_1 y_1^3 + \Lambda_2 y_1^2 y_2 + \Lambda_3 y_1 y_2^2 + \Lambda_4 y_2^3 \tag{2.30}$$

$$g_{12} = \Lambda_5 y_1^3 + \Lambda_6 y_1^2 y_2 + \Lambda_7 y_1 y_2^2 + \Lambda_8 y_2^3 \tag{2.31}$$

Substituting (2.28)–(2.31) into (2.27) and equating the coefficients of y_1^3, $y_1^2 y_2$, $y_1 y_2^2$, and y_2^3 on both sides of each row, we obtain (2.18) where

$$B = \begin{bmatrix} 2\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1+\lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\lambda_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3\lambda_2-\lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3\lambda_1-\lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\lambda_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1+\lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\lambda_2 \end{bmatrix} \tag{2.32}$$

Again, the eigenvalues $\lambda_{m,i}$ of the matrix B are related to the eigenvalues λ_1 and λ_2 of J and hence A as in (2.20), but here

$$m_1 + m_2 = 3 \quad \text{and} \quad m_1\,, \quad m_2 = 0, 1, 2\,, \quad \text{and} \quad 3$$

It follows from (2.18) and (2.32) that if none of the eigenvalues of the matrix B is zero, we can set $\boldsymbol{\Lambda} = 0$, solve for the Γ_m, and obtain

$$\Gamma_1 = \frac{\alpha_1}{2\lambda_1}\,, \quad \Gamma_2 = \frac{\alpha_2}{\lambda_1+\lambda_2}\,, \quad \Gamma_3 = \frac{\alpha_3}{2\lambda_2}\,, \quad \Gamma_4 = \frac{\alpha_4}{3\lambda_2-\lambda_1}\,,$$
$$\Gamma_5 = \frac{\alpha_5}{3\lambda_1-\lambda_2}\,, \quad \Gamma_6 = \frac{\alpha_6}{2\lambda_1}\,, \quad \Gamma_7 = \frac{\alpha_7}{\lambda_1+\lambda_2}\,, \quad \Gamma_8 = \frac{\alpha_8}{2\lambda_2} \tag{2.33}$$

With the choices in (2.33), all of the nonlinear terms are eliminated from (2.26), resulting in the linear normal form

$$\dot{y} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} y \tag{2.34}$$

It follows from (2.33) that some of the Γ_m are unbounded or have small divisors if one or more of the eigenvalues of the matrix B vanish or nearly vanish; that is, if one of the following resonance conditions is satisfied:

$$\lambda_1 \approx 0\,, \quad \lambda_2 \approx 0\,, \quad \lambda_1+\lambda_2 \approx 0\,, \quad 3\lambda_2-\lambda_1 \approx 0\,, \quad 3\lambda_1-\lambda_2 \approx 0$$

When $\lambda_1 \approx 0$, we let $\Lambda_1 = \alpha_1$ and $\Lambda_6 = \alpha_6$ to avoid the singularities or small-divisor terms, set the remaining Λ_m equal to zero, and choose the Γ_m, except Γ_1 and Γ_6, as in (2.33). Then, the normal form of (2.26) is

$$\dot{y} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} y + \epsilon \begin{bmatrix} \alpha_1 y_1^3 \\ \alpha_6 y_1^2 y_2 \end{bmatrix} \tag{2.35}$$

When $\lambda_2 \approx 0$, we let $\Lambda_3 = \alpha_3$ and $\Lambda_8 = \alpha_8$ to avoid the singularities or small-divisor terms, set the remaining Λ_m equal to zero, and choose the Γ_m, except Γ_3 and Γ_8 as in (2.33). Then, the normal form of (2.26) in this case becomes

$$\dot{y} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} y + \epsilon \begin{bmatrix} \alpha_3 y_1 y_2^2 \\ \alpha_8 y_2^3 \end{bmatrix} \tag{2.36}$$

When $\lambda_1 + \lambda_2 \approx 0$, we let $\Lambda_2 = \alpha_2$ and $\Lambda_7 = \alpha_7$ to avoid the singularities or small-divisor terms, set the remaining Λ_m equal to zero, and choose the Γ_m, except Γ_2 and Γ_7 as in (2.33). Then, the normal form of (2.26) is

$$\dot{y} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} y + \epsilon \begin{bmatrix} \alpha_2 y_1^2 y_2 \\ \alpha_7 y_1 y_2^2 \end{bmatrix} \tag{2.37}$$

When $3\lambda_2 - \lambda_1 \approx 0$, we let $\Lambda_4 = \alpha_4$ to avoid the singularity or small-divisor term, set the remaining Λ_m equal to zero, and choose the Γ_m, except Γ_4 as in (2.33). In this case, the normal form of (2.26) is

$$\dot{y} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} y + \epsilon \begin{bmatrix} \alpha_4 y_2^3 \\ 0 \end{bmatrix} \tag{2.38}$$

Similarly, the normal form of (2.26) when $3\lambda_1 - \lambda_2 \approx 0$ is

$$\dot{y} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} y + \epsilon \begin{bmatrix} 0 \\ \alpha_5 y_1^3 \end{bmatrix} \tag{2.39}$$

Again, the matrix B can be viewed as a map of the $\Re^8$ space into itself. Moreover, the $\boldsymbol{e}_i$, where $\boldsymbol{e}_i$ is an eight-dimensional vector with all its components being zero except the ith component being one, form a basis for $\Re^8$. The subset of eigenvectors of B with nonzero eigenvalues form a basis for the image X of $\Re^8$ under B. Consequently, the components of α in X can be eliminated by a proper choice of Γ, whereas the components of α in the subspace X^c complementary to X cannot be removed by any choice of Γ. The latter are called resonance terms; they are spanned by the eigenvectors of B with zero eigenvalues. The near-resonance terms are spanned by the eigenvectors of B with near-zero eigenvalues.

2.3
A Two-Dimensional System with a Nonsemisimple Linear Form

In this section, we consider a two-dimensional system whose linear part has a non-semisimple Jordan form; that is, we consider

$$\dot{x} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} x + \epsilon \begin{bmatrix} \alpha_1 x_1^2 + \alpha_2 x_1 x_2 + \alpha_3 x_2^2 \\ \alpha_4 x_1^2 + \alpha_5 x_1 x_2 + \alpha_6 x_2^2 \end{bmatrix} \tag{2.40}$$

Substituting (2.40) into (2.8) yields

$$\begin{bmatrix} g_{11} \\ g_{12} \end{bmatrix} + \begin{bmatrix} \dfrac{\partial h_{11}}{\partial y_1} & \dfrac{\partial h_{11}}{\partial y_2} \\ \dfrac{\partial h_{12}}{\partial y_1} & \dfrac{\partial h_{12}}{\partial y_2} \end{bmatrix} \begin{bmatrix} \lambda y_1 + y_2 \\ \lambda y_2 \end{bmatrix} - \begin{bmatrix} \lambda h_{11} + h_{12} \\ \lambda h_{12} \end{bmatrix}$$
$$= \begin{bmatrix} \alpha_1 y_1^2 + \alpha_2 y_1 y_2 + \alpha_3 y_2^2 \\ \alpha_4 y_1^2 + \alpha_5 y_1 y_2 + \alpha_6 y_2^2 \end{bmatrix} \tag{2.41}$$

The form of the right-hand side of (2.41) suggests seeking the h_{1m} and g_{1m} in the form (2.12)–(2.15). Substituting (2.12)–(2.15) into (2.41) and equating the coefficients of y_1^2, $y_1 y_2$, and y_2^2 on both sides of each row, we obtain (2.18), where

$$B = \begin{bmatrix} \lambda & 0 & 0 & -1 & 0 & 0 \\ 2 & \lambda & 0 & 0 & -1 & 0 \\ 0 & 1 & \lambda & 0 & 0 & -1 \\ 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 2 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 1 & \lambda \end{bmatrix} \tag{2.42}$$

The eigenvalues of B are λ with a multiplicity of six. Hence, as long as $\lambda \neq 0$, one can solve for $\boldsymbol{\Gamma}$ for any $\boldsymbol{\alpha}$. Consequently, we put $\boldsymbol{\Lambda} = 0$, solve (2.18) for $\boldsymbol{\Gamma}$ in terms of $\boldsymbol{\alpha}$, and obtain

$$\Gamma_1 = \frac{\alpha_4 + \lambda\alpha_1}{\lambda^2}, \quad \Gamma_2 = \frac{\lambda\alpha_5 - 4\alpha_4 + \lambda^2\alpha_2 - 2\lambda\alpha_1}{\lambda^3},$$
$$\Gamma_6 = \frac{\lambda^2\alpha_6 - \lambda\alpha_5 + 2\alpha_4}{\lambda^3},$$
$$\Gamma_3 = \frac{\lambda^2\alpha_6 - 2\lambda\alpha_5 + 6\alpha_4 + \lambda^3\alpha_3 - \lambda^2\alpha_2 + 2\lambda\alpha_1}{\lambda^4},$$
$$\Gamma_4 = \frac{\alpha_4}{\lambda}, \quad \Gamma_5 = \frac{\lambda\alpha_5 - 2\alpha_4}{\lambda^2}$$

With these choices for the Γ_m, all of the nonlinear terms are eliminated from (2.40), resulting in the linear normal form

$$\dot{y} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} y \tag{2.43}$$

Next, we consider (2.40) with cubic, rather than quadratic, nonlinearities; that is,

$$\dot{x} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} x + \epsilon \begin{bmatrix} \alpha_1 x_1^3 + \alpha_2 x_1^2 x_2 + \alpha_3 x_1 x_2^2 + \alpha_4 x_2^3 \\ \alpha_5 x_1^3 + \alpha_6 x_1^2 x_2 + \alpha_7 x_1 x_2^2 + \alpha_8 x_2^3 \end{bmatrix} \tag{2.44}$$

Substituting (2.44) into (2.8) yields

$$\begin{aligned} &\begin{bmatrix} g_{11} \\ g_{12} \end{bmatrix} + \begin{bmatrix} \dfrac{\partial h_{11}}{\partial y_1} & \dfrac{\partial h_{11}}{\partial y_2} \\ \dfrac{\partial h_{12}}{\partial y_1} & \dfrac{\partial h_{12}}{\partial y_2} \end{bmatrix} \begin{bmatrix} \lambda y_1 + y_2 \\ \lambda y_2 \end{bmatrix} - \begin{bmatrix} \lambda h_{11} + h_{12} \\ \lambda h_{12} \end{bmatrix} \\ &= \begin{bmatrix} \alpha_1 y_1^3 + \alpha_2 y_1^2 y_2 + \alpha_3 y_1 y_2^2 + \alpha_4 y_2^3 \\ \alpha_5 y_1^3 + \alpha_6 y_1^2 y_2 + \alpha_7 y_1 y_2^2 + \alpha_8 y_2^3 \end{bmatrix} \end{aligned} \tag{2.45}$$

The form of the right-hand side of (2.45) suggests seeking the h_{1m} and g_{1m} in the form (2.28)–(2.31). Substituting (2.28)–(2.31) into (2.45) and equating the coefficients of y_1^3, $y_1^2 y_2$, $y_1 y_2^2$, and y_2^3 on both sides of each row, we obtain (2.18), where

$$B = \begin{bmatrix} 2\lambda & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 3 & 2\lambda & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 2\lambda & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2\lambda & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 2\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 2\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2\lambda \end{bmatrix} \tag{2.46}$$

The eigenvalues of B are 2λ with a multiplicity of eight. Hence, as long as $\lambda \neq 0$, B is invertible and one can solve (2.18) for Γ, irrespective of the value of α. Consequently, we let $\Lambda = 0$ and solve (2.18) for Γ in terms of α, thereby eliminating all of the nonlinear terms from (2.44) and obtaining the linear normal form given by (2.43).

We note that the eigenvalues of the matrices (2.42) and (2.46) are related to the eigenvalue λ of the linear parts of (2.40) and (2.44) as in (2.20), where $m_1 + m_2 = r$ and r is the degree of the polynomial on the right-hand side of either (2.40) or (2.44). Thus, when $\lambda_1 = \lambda_2 = \lambda$ and $m_1 + m_2 = r$, (2.20) yields the eigenvalues $(r-1)\lambda$ with multiplicity six when $r = 2$ and multiplicity eight when $r = 3$, in agreement with the detailed calculations presented in this section.

2.4 An *n*-Dimensional System with Diagonal Linear Part

Next, we consider the following n-dimensional system with homogeneous polynomial nonlinearity of degree r:

$$\dot{x} = Jx + f_r(x) \tag{2.47}$$

where J is a diagonal matrix with the entries $\lambda_1, \lambda_2 \ldots, \lambda_n$, and $\boldsymbol{x}=(x_1, x_2, \ldots, x_n)^{\mathrm{T}}$. Here, $\boldsymbol{f}_r(\boldsymbol{x})$ is a real vector-valued function whose components are homogeneous polynomials of degree r. In other words, $\boldsymbol{f}_r(\boldsymbol{x})$ belongs to the space H_r, which is spanned by the vector-valued monomials

$$\boldsymbol{x}^m \boldsymbol{e}_i = x_1^{m_1} x_2^{m_2} \cdots x_n^{m_2} \boldsymbol{e}_i$$

where

$$\boldsymbol{m} = (m_1, m_2, \ldots, m_n) \ ; \quad m_i = 0, 1, 2, \ldots, r \ ; \quad m_1 + m_2 + \ldots + m_n = r$$

and $\boldsymbol{e}_i$ is a column vector whose n components are zero except the ith component is one.

Substituting (2.47) into (2.8) yields

$$\mathcal{L}_J \left[\boldsymbol{h}_1(\boldsymbol{y})\right] = D(\boldsymbol{h}_1(\boldsymbol{y})) J\boldsymbol{y} - J\boldsymbol{h}_1(\boldsymbol{y}) = \boldsymbol{f}_r(\boldsymbol{y}) - \boldsymbol{g}_1(\boldsymbol{y}) \tag{2.48}$$

where the operator $\mathcal{L}_J$ is known as the Lie or Poisson bracket of the vector fields $J\boldsymbol{y}$ and $\boldsymbol{h}_1(\boldsymbol{y})$. The Lie bracket $\mathcal{L}_J$ linearly maps H_r into H_r. To determine the eigenvalues and eigenvectors of $\mathcal{L}_J$, we note that

$$\begin{aligned}
\mathcal{L}_J \left(\boldsymbol{y}^m \boldsymbol{e}_i\right) &= D\left(\boldsymbol{y}^m \boldsymbol{e}_i\right) J\boldsymbol{y} - J\left(\boldsymbol{y}^m \boldsymbol{e}_i\right) \\
&= \boldsymbol{y}^m \begin{bmatrix} 0 & 0 & 0 & \bullet & \bullet & \bullet & 0 & 0 \\ 0 & 0 & 0 & \bullet & \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \frac{m_1}{y_1} & \frac{m_2}{y_2} & \frac{m_3}{y_3} & \bullet & \bullet & \bullet & \bullet & \frac{m_n}{y_n} \\ 0 & 0 & 0 & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & 0 & \bullet \\ 0 & 0 & 0 & \bullet & \bullet & \bullet & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 y_1 \\ \lambda_2 y_2 \\ \bullet \\ \lambda_i y_i \\ \bullet \\ \bullet \\ \lambda_n y_n \end{bmatrix} - \boldsymbol{y}^m \lambda_i \boldsymbol{e}_i \\
&= \left[(m_1\lambda_1 + m_2\lambda_2 + \cdots + m_n\lambda_n) - \lambda_i\right] y^{\boldsymbol{m}} \boldsymbol{e}_i \\
&= (\boldsymbol{m} \bullet \boldsymbol{\lambda} - \lambda_i)\, y^{\boldsymbol{m}} \boldsymbol{e}_i
\end{aligned} \tag{2.49}$$

Hence, the eigenvectors of $\mathcal{L}_J$ are $\boldsymbol{y}^m \boldsymbol{e}_i$ corresponding to the eigenvalues

$$\lambda_{\boldsymbol{m},i} = \boldsymbol{m} \bullet \boldsymbol{\lambda} - \lambda_i = m_1\lambda_1 + m_2\lambda_2 + \cdots + m_n\lambda_n - \lambda_i \tag{2.50}$$

Consequently, the subset of the eigenvectors of $\mathcal{L}_J$ with nonzero eigenvalues forms a basis for the image X_r of H_r under $\mathcal{L}_J$. Hence, the components of $\boldsymbol{f}_r(\boldsymbol{y})$ lying in X_r can be eliminated by a proper choice of $\boldsymbol{h}_1(\boldsymbol{y})$ and the components of $\boldsymbol{f}_r(\boldsymbol{y})$ not lying in X_r cannot be eliminated. The latter are called resonance terms; they are spanned by the eigenvectors of $\mathcal{L}_J$ with zero eigenvalues. The components of $\boldsymbol{f}_r(\boldsymbol{y})$ lying in the subspace spanned by the eigenvectors of $\mathcal{L}_J$ with near-zero eigenvalues are called near-resonance terms.

Considering (2.10), we note that $r = 2$ and the eigenvectors of $\mathcal{L}_J$ are $y_1^{m1} y_2^{m2} \boldsymbol{e}_i$, where $m_1 + m_2 = 2$ and $m_i = 0, 1$, and 2, with the eigenvalues

$$\lambda_{\boldsymbol{m},j} = m_1\lambda_1 + m_2\lambda_2 - \lambda_i$$

Thus, putting $(m_1, m_2) = (0, 2), (1, 1)$, and $(2, 0)$, we obtain the eigenvalues $2\lambda_2 - \lambda_1, \lambda_2$, and λ_1 when $i = 1$ and λ_2, λ_1, and $2\lambda_1 - \lambda_2$ when $i = 2$, in agreement with those obtained in the preceding section. Thus, when none of the $\lambda_{\boldsymbol{m},i}$ is zero, $\boldsymbol{h}_1$ can be chosen to eliminate all of the nonlinear terms in (2.10), thereby yielding the linear normal form $\dot{\boldsymbol{y}} = J\boldsymbol{y}$.

When $2\lambda_2 - \lambda_1 \approx 0$, all of the nonlinear terms can be eliminated except $\alpha_3 y_2^2 \boldsymbol{e}_1$, which is spanned by the eigenvector of $\mathcal{L}_J$ corresponding to the eigenvalue

$$2\lambda_2 - \lambda_1 = (0)\lambda_1 + (2)\lambda_2 - \lambda_1$$

Hence, the normal form of (2.10) is given by (2.22).

When $2\lambda_1 - \lambda_2 \approx 0$, all of the nonlinear terms in (2.10) can be eliminated except $\alpha_4 y_1^2 \boldsymbol{e}_2$, which is spanned by the eigenvector of $\mathcal{L}_J$ corresponding to the eigenvalue

$$2\lambda_1 - \lambda_2 = (2)\lambda_1 + (0)\lambda_2 - \lambda_2$$

Consequently, the normal form (2.10) is given by (2.23).

When $\lambda_1 \approx 0$, all of the nonlinear terms in (2.10) can be eliminated except $\alpha_1 y_1^2 \boldsymbol{e}_1$ and $\alpha_5 y_1 y_2 \boldsymbol{e}_2$, which are spanned by the eigenvectors of $\mathcal{L}_J$ corresponding to the eigenvalue

$$\lambda_1 = (2)\lambda_1 + (0)\lambda_2 - \lambda_1 = (1)\lambda_1 + (1)\lambda_2 - \lambda_2$$

Consequently, the normal form of (2.10) is given by (2.24).

Finally, when $\lambda_2 \approx 0$, all of the nonlinear terms in (2.10) can be eliminated except $\alpha_2 y_1 y_2 \boldsymbol{e}_1$ and $\alpha_6 y_2^2 \boldsymbol{e}_2$, which are spanned by the eigenvectors of $\mathcal{L}_J$ corresponding to the eigenvalue

$$\lambda_2 = (1)\lambda_1 + (1)\lambda_2 - \lambda_1 = (0)\lambda_1 + (2)\lambda_2 - \lambda_2$$

Consequently, the normal form of (2.10) is given by (2.25).

2.5
A Two-Dimensional System with Purely Imaginary Eigenvalues

As a first example, we consider the two-dimensional system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \boldsymbol{x} + \epsilon \begin{bmatrix} \alpha_1 x_1^3 + \alpha_2 x_1^2 x_2 + \alpha_3 x_1 x_2^2 + \alpha_4 x_2^3 \\ \alpha_5 x_1^3 + \alpha_6 x_1^2 x_2 + \alpha_7 x_1 x_2^2 + \alpha_8 x_2^3 \end{bmatrix} \tag{2.51}$$

We note that the nonzero elements in the matrix J are +1 and -1. If they have other values, one can introduce time and coordinate transformations so that J has the form in (2.51). Equation 2.51 can conveniently be transformed into a single complex-valued equation using the transformation

$$x_1 = \zeta + \bar{\zeta} \quad \text{and} \quad x_2 = i(\zeta - \bar{\zeta}) \tag{2.52}$$

and obtaining

$$\dot{\zeta} = i\zeta + \tfrac{1}{2}\epsilon\Big[(\alpha_1 - i\alpha_5)\left(\zeta + \bar{\zeta}\right)^3 + i\left(\alpha_2 - i\alpha_6\right)\left(\zeta + \bar{\zeta}\right)^2\left(\zeta - \bar{\zeta}\right) - \left(\alpha_3 - i\alpha_7\right)\left(\zeta + \bar{\zeta}\right)\left(\zeta - \bar{\zeta}\right)^2 - i\left(\alpha_4 - i\alpha_8\right)\left(\zeta - \bar{\zeta}\right)^3\Big] \tag{2.53}$$

Next, we treat both (2.51) and (2.53) by using both the method of normal forms and the method of multiple scales.

Carrying out a straightforward expansion by putting $\zeta = Ae^{it}$ at $O(1)$, one finds that the term proportional to $\zeta^2\bar{\zeta}$ produces a secular term proportional to $A^2\bar{A}te^{it}$, whereas the remaining terms (i.e., those proportional to ζ^3, $\zeta\bar{\zeta}^2$, and $\bar{\zeta}^3$) do not produce secular terms. Hence, only the term proportional to $\zeta^2\bar{\zeta}$ is a resonance term.

2.5.1 The Method of Normal Forms

We start by treating (2.53). Thus, to first order, we let

$$\zeta = \eta + \epsilon h(\eta, \bar{\eta}) + \cdots \tag{2.54}$$

$$\dot{\eta} = i\eta + \epsilon g(\eta, \bar{\eta}) + \cdots \tag{2.55}$$

Substituting (2.54) and (2.55) into (2.53) and equating the coefficient of ϵ on both sides, we obtain

$$g + i\left(\frac{\partial h}{\partial \eta}\eta - \frac{\partial h}{\partial \bar{\eta}}\bar{\eta} - h\right) = \frac{1}{2}\left[(\alpha_1 - i\alpha_5)(\eta + \bar{\eta})^3 - i(\alpha_4 - i\alpha_8)(\eta - \bar{\eta})^3 + i(\alpha_2 - i\alpha_6)(\eta + \bar{\eta})^2(\eta - \bar{\eta}) - (\alpha_3 - i\alpha_7)(\eta + \bar{\eta})(\eta - \bar{\eta})^2\right] \tag{2.56}$$

Next, we choose h to eliminate the nonresonance terms, thereby leaving g with the resonance terms. The form of the terms on the right-hand side of (2.56) suggests choosing h in the form

$$h = \Gamma_1\eta^3 + \Gamma_2\eta^2\bar{\eta} + \Gamma_3\eta\bar{\eta}^2 + \Gamma_4\bar{\eta}^3 \tag{2.57}$$

Substituting (2.57) into (2.56) and rearranging the result, we obtain

$$\begin{aligned} & g - 4(a + ib)\eta^2\bar{\eta} \\ & \quad + \left[2i\Gamma_1 - \tfrac{1}{2}(\alpha_1 - \alpha_3 + \alpha_6 - \alpha_8) - \tfrac{1}{2}i(\alpha_2 - \alpha_4 - \alpha_5 + \alpha_7)\right]\eta^3 \\ & \quad - \left[2i\Gamma_3 + \tfrac{1}{2}(3\alpha_1 + \alpha_3 - \alpha_6 - 3\alpha_8) - \tfrac{1}{2}i(\alpha_2 + 3\alpha_4 + 3\alpha_5 + \alpha_7)\right]\eta\bar{\eta}^2 \\ & \quad - \left[4i\Gamma_4 + \tfrac{1}{2}(\alpha_1 - \alpha_3 - \alpha_6 + \alpha_8) - \tfrac{1}{2}i(\alpha_2 - \alpha_4 + \alpha_5 - \alpha_7)\right]\bar{\eta}^3 = 0 \end{aligned} \tag{2.58}$$

where

$$\begin{aligned} 8a &= 3\alpha_1 + \alpha_3 + \alpha_6 + 3\alpha_8 \\ 8b &= \alpha_2 + 3\alpha_4 - 3\alpha_5 - \alpha_7 \end{aligned} \tag{2.59}$$

Equation 2.58 is independent of Γ_2, indicating that $\eta^2\bar{\eta}$ is a resonance term. Consequently, we can choose Γ_1, Γ_3, and Γ_4 to eliminate the terms proportional to $\eta^3, \eta\bar{\eta}^2$, and $\bar{\eta}^3$, thereby leaving g with the resonance terms; that is

$$g = 4(a + ib)\eta^2\bar{\eta} \tag{2.60}$$

We note that one does not need to go through the details of determining h to find out the resonance terms. They can be found by just inspecting either (2.53) or (2.56) to determine the terms that produce a frequency near the unperturbed frequency, unity in this case. Because η has approximately the unperturbed frequency 1, then $\eta^3, \eta^2\bar{\eta}, \eta\bar{\eta}^2$, and $\bar{\eta}^3$ have approximately the frequencies 3, 1, -1, and -3, respectively. Hence, only the term $\eta^2\bar{\eta}$ is a resonance term. Alternatively, to the first approximation $\eta = e^{it}$, and hence the terms $\eta^3, \eta^2\bar{\eta}, \eta\bar{\eta}^2$, and $\bar{\eta}^3$ are proportional to e^{3it}, e^{it}, e^{-it}, and e^{-3it}. Because only the term proportional to e^{it} produces a secular term, only $\eta^2\bar{\eta}$ is a resonance term.

Substituting (2.60) into (2.55) yields, to first order, the normal form

$$\dot{\eta} = i\eta + 4\epsilon\,(a + ib)\,\eta^2\bar{\eta} \tag{2.61}$$

The normal form (2.61) has a simple representation when expressed in polar coordinates; that is,

$$\eta = \tfrac{1}{2} r e^{i\beta} \tag{2.62}$$

Substituting (2.62) into (2.61) and separating real and imaginary parts, we obtain

$$\dot{r} = \epsilon a r^3 \tag{2.63}$$

$$\dot{\beta} = 1 + \epsilon b r^2 \tag{2.64}$$

in which the equation describing the amplitude r is uncoupled from that describing the phase β. On the other hand, (2.61) has a more complicated form when expressed in Cartesian coordinates; that is,

$$\eta = \tfrac{1}{2}\,(y_1 - i y_2) \tag{2.65}$$

Substituting (2.65) into (2.61) and separating real and imaginary parts, we obtain

$$\dot{y}_1 = y_2 + \epsilon\,(a y_1 + b y_2)\left(y_1^2 + y_2^2\right) \tag{2.66}$$

$$\dot{y}_2 = -y_1 + \epsilon\,(a y_2 - b y_1)\left(y_1^2 + y_2^2\right) \tag{2.67}$$

Next, we treat (2.51) in which real rather than complex variables are used. Substituting (2.51) into (2.8) yields

$$\begin{bmatrix} g_{11} \\ g_{12} \end{bmatrix} + \begin{bmatrix} \dfrac{\partial h_{11}}{\partial y_1} & \dfrac{\partial h_{11}}{\partial y_2} \\ \dfrac{\partial h_{12}}{\partial y_1} & \dfrac{\partial h_{12}}{\partial y_2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \end{bmatrix}$$
$$= \begin{bmatrix} \alpha_1 y_1^3 + \alpha_2 y_1^2 y_2 + \alpha_3 y_1 y_2^2 + \alpha_4 y_2^3 \\ \alpha_5 y_1^3 + \alpha_6 y_1^2 y_2 + \alpha_7 y_1 y_2^2 + \alpha_8 y_2^3 \end{bmatrix} \tag{2.68}$$

where (h_{11}, h_{12}) and (g_{11}, g_{12}) are the components of $\boldsymbol{h}$ and $\mathbf{g}$. The right-hand side of (2.68) suggests seeking the h_{1m} and g_{1m} as in (2.28)–(2.31). Substituting (2.28)–(2.31) into (2.68), we have

$$\begin{aligned} &\Lambda_1 y_1^3 + \Lambda_2 y_1^2 y_2 + \Lambda_3 y_1 y_2^2 + \Lambda_4 y_2^3 + \left(3\Gamma_1 y_1^2 + 2\Gamma_2 y_1 y_2 + \Gamma_3 y_2^2\right) y_2 \\ &\quad - \left(\Gamma_2 y_1^2 + 2\Gamma_3 y_1 y_2 + 3\Gamma_4 y_2^2\right) y_1 \\ &\quad - \left(\Gamma_5 y_1^3 + \Gamma_6 y_1^2 y_2 + \Gamma_7 y_1 y_2^2 + \Gamma_8 y_2^3\right) \\ &= \alpha_1 y_1^3 + \alpha_2 y_1^2 y_2 + \alpha_3 y_1 y_2^2 + \alpha_4 y_2^3 \end{aligned} \tag{2.69}$$

$$\begin{aligned} &\Lambda_5 y_1^3 + \Lambda_6 y_1^2 y_2 + \Lambda_7 y_1 y_2^2 + \Lambda_8 y_2^3 + \left(3\Gamma_5 y_1^2 + 2\Gamma_6 y_1 y_2 + \Gamma_7 y_2^2\right) y_2 \\ &\quad - \left(\Gamma_6 y_1^2 + 2\Gamma_7 y_1 y_2 + 3\Gamma_8 y_2^2\right) y_1 \\ &\quad + \left(\Gamma_1 y_1^3 + \Gamma_2 y_1^2 y_2 + \Gamma_3 y_1 y_2^2 + \Gamma_4 y_2^3\right) \\ &= \alpha_5 y_1^3 + \alpha_6 y_1^2 y_2 + \alpha_7 y_1 y_2^2 + \alpha_8 y_2^3 \end{aligned} \tag{2.70}$$

Equating the coefficients of $y_1^3, y_1^2 y_2, y_1 y_2^2$, and y_2^3 on both sides of (2.69) and (2.70), we obtain (2.18), where $\boldsymbol{\Gamma}, \boldsymbol{\alpha}$, and $\boldsymbol{\Lambda}$ are column vectors with the components Γ_m, α_m, and Λ_m, respectively, and

$$B = \begin{bmatrix} 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 3 & 0 & -2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & -3 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{2.71}$$

Because the eigenvalues of the linear part of (2.51) are $\lambda_1 = i$ and $\lambda_2 = -i$, which are distinct, this linear part can be transformed into a diagonal form, by using a complex-valued transformation. Consequently, the eigenvalues of the Lie bracket $\mathcal{L}_J$ (i.e., the eigenvalues of B) are related to λ_1 and λ_2 by the relation (2.20), where $(m_1, m_2) = (0, 3), (1, 2), (2, 1)$, and $(3, 0)$. Hence, the eigenvalues of B are $2i, 2i, -2i, -2i, 4i, -4i, 0$, and 0, and one cannot solve (2.18) for $\boldsymbol{\Gamma}$ to remove all of the nonlinear terms. The resonance terms, the terms that cannot be removed by any choice of $\boldsymbol{\Gamma}$, are spanned by the eigenvectors of B corresponding to the two zero eigenvalues. In other words, the resonance terms, and hence $\boldsymbol{\Lambda}$, belong to (or are members of) the null space of B; that is, the space spanned by the nontrivial solutions of

$$B\boldsymbol{\Gamma} = 0$$

Hence,

$$\boldsymbol{\Lambda} = c_1(1,0,1,0,0,1,0,1)^{\mathrm{T}} + c_2(0,1,0,1,-1,0,-1,0)^{\mathrm{T}} \tag{2.72}$$

where c_1 and c_2 are constants and $\boldsymbol{b}^{\mathrm{T}}$ denotes the transpose of $\boldsymbol{b}$. Consequently, to first order, the normal form of (2.51) is

$$\dot{y}_1 = y_2 + \epsilon\,(c_1 y_1 + c_2 y_2)\left(y_1^2 + y_2^2\right) \tag{2.73}$$

$$\dot{y}_2 = -y_1 + \epsilon\,(c_1 y_2 - c_2 y_1)\left(y_1^2 + y_2^2\right) \tag{2.74}$$

To determine c_1 and c_2, we have two alternatives. First, we determine the image (range) X of $\Re^8$ under B and then determine the subspace X^c complementary to X. The projection $\boldsymbol{\alpha}^c$ of $\boldsymbol{\alpha}$ on X^c will be the part that cannot be eliminated by any choice of $\boldsymbol{\Gamma}$ (i.e., resonance terms) and hence $\boldsymbol{\alpha}^c$ will remain in the normal form. Second, we note that, because B is singular (it has a rank of 6 and a null space of rank 2), (2.18) has a solution if and only if $\boldsymbol{\alpha} - \boldsymbol{\Lambda}$ is orthogonal to the two linearly independent solution vectors of the adjoint problem

$$B^{\mathrm{T}}\boldsymbol{\Gamma} = 0 \tag{2.75}$$

where B^{T} is the transpose of B, in which case $\boldsymbol{\Gamma}$ is not unique. Equation 2.75 has the following two linearly independent nontrivial solutions:

$$(3,0,1,0,0,1,0,3)^{\mathrm{T}} \quad \text{and} \quad (0,1,0,3,-3,0,-1,0)^{\mathrm{T}} \tag{2.76}$$

Consequently, (2.18) has a solution if and only if $\boldsymbol{\alpha} - \boldsymbol{\Lambda}$ is orthogonal to each of the vectors in (2.76), which upon using (2.72) yields

$$\begin{aligned} 8c_1 &= 3\alpha_1 + \alpha_3 + \alpha_6 + 3\alpha_8 \\ 8c_2 &= \alpha_2 + 3\alpha_4 - 3\alpha_5 - \alpha_7 \end{aligned} \tag{2.77}$$

Comparing (2.59) and (2.77), we find that $c_1 = a$ and $c_2 = b$, and hence the normal form (2.73) and (2.74) obtained by attacking the two real-valued first-order equations is the same as the normal form (2.66) and (2.67) obtained by attacking the single complex-valued equation. However, the algebra involved in attacking the complex-valued equation is much less than that involved in attacking the two real-valued equations.

2.5.2 The Method of Multiple Scales

We start by treating the complex-valued form (2.53) and seek a first-order uniform expansion in the form

$$\zeta = \zeta_0(T_0, T_1) + \epsilon\zeta_1(T_0, T_1) + \cdots \tag{2.78}$$

where $T_0 = t$ and $T_1 = \epsilon t$. Hence,

$$\frac{d}{dt} = D_0 + \epsilon D_1 + \cdots, \quad D_n = \frac{\partial}{\partial T_n} \tag{2.79}$$

Substituting (2.78) and (2.79) into (2.53) and equating coefficients of like powers of ϵ on both sides, we obtain

$$D_0\zeta_0 - i\zeta_0 = 0 \tag{2.80}$$

$$\begin{aligned} D_0\zeta_1 - i\zeta_1 = {} & -D_1\zeta_0 + \tfrac{1}{2}\Big[(\alpha_1 - i\alpha_5)\left(\zeta_0 + \bar{\zeta}_0\right)^3 - i(\alpha_4 - i\alpha_8)\left(\zeta_0 - \bar{\zeta}_0\right)^3 \\ & -(\alpha_3 - i\alpha_7)\left(\zeta_0 + \bar{\zeta}_0\right)\left(\zeta_0 - \bar{\zeta}_0\right)^2 + i(\alpha_2 - i\alpha_6)\left(\zeta_0 + \bar{\zeta}_0\right)^2\left(\zeta_0 - \bar{\zeta}_0\right)\Big] \end{aligned} \tag{2.81}$$

The solution of (2.80) can be expressed as

$$\zeta_0 = A(T_1)e^{iT_0} \tag{2.82}$$

where A is an arbitrary function of T_1 to this order; it is determined by eliminating the secular terms from ζ_1. Substituting (2.82) into (2.81) and eliminating the terms that produce secular terms in ζ_1, we obtain

$$D_1 A = 4(a + ib)A^2\bar{A} \tag{2.83}$$

where a and b are defined in (2.59). Letting $\eta = Ae^{it}$ in (2.61), we obtain (2.83) because $\epsilon D_1 A = \dot{A}$.

Next, we use the method of multiple scales to treat (2.51). We begin by seeking a first-order expansion in the form

$$x_1 = x_{11}(T_0, T_1) + \epsilon x_{12}(T_0, T_1) + \cdots \tag{2.84}$$

$$x_2 = x_{21}(T_0, T_1) + \epsilon x_{22}(T_0, T_1) + \cdots \tag{2.85}$$

Substituting (2.84) and (2.85) into (2.51), using (2.79), and equating coefficients of like powers of ϵ on both sides, we obtain

$$D_0 x_{11} - x_{21} = 0 \tag{2.86a}$$

$$D_0 x_{21} + x_{11} = 0 \tag{2.86b}$$

$$D_0 x_{12} - x_{22} = -D_1 x_{11} + \alpha_1 x_{11}^3 + \alpha_2 x_{11}^2 x_{21} + \alpha_3 x_{11} x_{21}^2 + \alpha_4 x_{21}^3 \tag{2.87a}$$

$$D_0 x_{22} + x_{12} = -D_1 x_{21} + \alpha_5 x_{11}^3 + \alpha_6 x_{11}^2 x_{21} + \alpha_7 x_{11} x_{21}^2 + \alpha_8 x_{21}^3 \tag{2.87b}$$

The solution of (2.86) can be expressed as

$$x_{11} = A(T_1)e^{iT_0} + \text{cc} \quad \text{and} \quad x_{21} = iA(T_1)e^{iT_0} + \text{cc} \tag{2.88}$$

where cc stands for the complex conjugate of the preceding terms. Substituting (2.88) into (2.87) yields

$$D_0 x_{12} - x_{22} = -D_1 A e^{iT_0} + \left[3\alpha_1 + \alpha_3 + i(\alpha_2 + 3\alpha_4)\right] A^2 \bar{A} e^{iT_0} + \text{cc} + \text{NST} \tag{2.89}$$

$$D_0 x_{22} + x_{12} = -iD_1 A e^{iT_0} + \left[3\alpha_5 + \alpha_7 + i(\alpha_6 + 3\alpha_8)\right] A^2 \bar{A} e^{iT_0} + \text{cc} + \text{NST}\,. \tag{2.90}$$

Differentiating (2.90) once with respect to T_0 and using (2.89) to eliminate $D_0 x_{12}$ from the result, we obtain

$$D_0^2 x_{22} + x_{22} = 2D_1 A e^{iT_0} - 8\,(a + ib)\,A^2 \bar{A} e^{iT_0} + \text{cc} + \text{NST} \tag{2.91}$$

where a and b are defined in (2.59). Eliminating the terms that produce secular terms from (2.91) yields (2.83).

Again, the algebra involved in attacking the Cartesian real form of the problem is more than that involved in attacking the complex-valued form of the problem.

2.6
A Two-Dimensional System with Zero Eigenvalues

We consider a two-dimensional system having quadratic nonlinearities in the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \epsilon \begin{bmatrix} \alpha_1 x_1^2 + \alpha_2 x_1 x_2 + \alpha_3 x_2^2 \\ \alpha_4 x_1^2 + \alpha_5 x_1 x_2 + \alpha_6 x_2^2 \end{bmatrix} \tag{2.92}$$

Then, we consider the same system with cubic, instead of quadratic, nonlinearities. We note that (2.92) differs drastically from (2.51) in that the unperturbed part of (2.51) can be diagonalized, whereas the unperturbed part of (2.92) cannot be diagonalized.

Substituting (2.92) into (2.8) yields

$$\begin{bmatrix} g_{11} \\ g_{12} \end{bmatrix} + \begin{bmatrix} \dfrac{\partial h_{11}}{\partial y_1} & \dfrac{\partial h_{11}}{\partial y_2} \\ \dfrac{\partial h_{12}}{\partial y_1} & \dfrac{\partial h_{12}}{\partial y_2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \end{bmatrix} = \begin{bmatrix} \alpha_1 y_1^2 + \alpha_2 y_1 y_2 + \alpha_3 y_2^2 \\ \alpha_4 y_1^2 + \alpha_5 y_1 y_2 + \alpha_6 y_2^2 \end{bmatrix} \tag{2.93}$$

where (h_{11}, h_{12}) and (g_{11}, g_{12}) are the components of $\boldsymbol{h}_1$ and $\boldsymbol{g}_1$. The right-hand side of (2.93) suggests seeking the h_{1m} and g_{1m} as in (2.12)–(2.15). Substituting (2.12)–

(2.15) into (2.93) yields

$$\Lambda_1 y_1^2 + \Lambda_2 y_1 y_2 + \Lambda_3 y_2^2 + y_2\left(2\Gamma_1 y_1 + \Gamma_2 y_2\right) - \left(\Gamma_4 y_1^2 + \Gamma_5 y_1 y_2 + \Gamma_6 y_2^2\right)$$
$$= \alpha_1 y_1^2 + \alpha_2 y_1 y_2 + \alpha_3 y_2^2 \tag{2.94}$$

$$\Lambda_4 y_1^2 + \Lambda_5 y_1 y_2 + \Lambda_6 y_2^2 + y_2\left(2\Gamma_4 y_1 + \Gamma_5 y_2\right) = \alpha_4 y_1^2 + \alpha_5 y_1 y_2 + \alpha_6 y_2^2 \tag{2.95}$$

Equating the coefficients of $y_1^2, y_1 y_2$, and y_2^2 on both sides of (2.94) and (2.95), we obtain (2.18), where $\boldsymbol{\Gamma}, \boldsymbol{\alpha}$, and $\boldsymbol{\Lambda}$ are column vectors having the components Γ_m, α_m, and Λ_m, respectively, and

$$B = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 2 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{2.96}$$

Because the linear part of the system under consideration cannot be diagonalized, the resonance terms are not spanned by the null space of B. Since the matrix B is singular (it has a rank of 4), the system of (2.18) has a solution if and only if $\boldsymbol{\alpha} - \boldsymbol{\Lambda}$ is orthogonal to every left-hand vector of B or to every eigenvector of B^{T} with a zero eigenvalue, where B^{T} is the adjoint of B; that is, if and only if $\boldsymbol{\alpha} - \boldsymbol{\Lambda}$ is orthogonal to every solution of (2.75) where B is defined in (2.96). Because the nontrivial solutions of (2.75) and (2.96) in this case are

$$(0,0,0,1,0,0)^{\mathrm{T}} \quad \text{and} \quad (2,0,0,0,1,0)^{\mathrm{T}}$$

(2.18) has a solution if and only if

$$\alpha_4 - \Lambda_4 = 0 \quad \text{and} \quad 2\left(\alpha_1 - \Lambda_1\right) + \alpha_5 - \Lambda_5 = 0$$

or

$$\Lambda_4 = \alpha_4 \quad \text{and} \quad 2\Lambda_1 + \Lambda_5 = 2\alpha_1 + \alpha_5 \tag{2.97}$$

With conditions (2.97) satisfied, one can solve for $\boldsymbol{\Gamma}$ for all values of the α_m, irrespective of the values of Λ_2, Λ_3, and Λ_6, which can be set equal to zero. Consequently, the normal form of (2.92) is

$$\dot{y}_1 = y_2 + \epsilon \Lambda_1 y_1^2 \tag{2.98}$$

$$\dot{y}_2 = \epsilon\left(\alpha_4 y_1^2 + \Lambda_5 y_1 y_2\right) \tag{2.99}$$

where Λ_1 and Λ_5 satisfy (2.97). Hence, the normal form is not unique because there is no unique solution for (2.97).

As mentioned earlier, because the linear part of the system under consideration cannot be diagonalized, the resonance terms are not spanned by the null space of B in this case, and hence this notion cannot be used to uniquely specify the normal form. Moreover, attempts to use this concept might lead to contradictions. For example, we suppose that $\boldsymbol{\Lambda}$ belongs to the null space of B; that is, the space spanned by the nontrivial solutions of

$$B\Gamma = 0 \tag{2.100}$$

Then, one finds that

$$\Lambda = c_1(0, 0, 1, 0, 0, 0)^{\mathrm{T}} + c_2(0, 1, 0, 0, 0, 1)^{\mathrm{T}} \tag{2.101}$$

which leads to the normal form

$$\dot{y}_1 = y_2 + \epsilon\left(c_2 y_1 y_2 + c_1 y_2^2\right) \tag{2.102}$$

$$\dot{y}_2 = \epsilon c_2 y_2^2 \tag{2.103}$$

The form in (2.102) and (2.103) is completely different from that given by (2.98) and (2.99). Moreover, using the expression (2.101) for $\boldsymbol{\Lambda}$ and requiring $\boldsymbol{\alpha} - \boldsymbol{\Lambda}$ to be orthogonal to the nontrivial solutions of (2.75), one obtains

$$\alpha_4 = 0 \quad \text{and} \quad 2\alpha_1 + \alpha_5 = 0$$

which, in general, are not satisfied. Consequently, the form given by (2.102) and (2.103) is wrong.

Next, we consider system (2.92) with cubic, rather than quadratic, nonlinearities; that is,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \epsilon\begin{bmatrix}\alpha_1 x_1^3 + \alpha_2 x_1^2 x_2 + \alpha_3 x_1 x_2^2 + \alpha_4 x_2^3\\ \alpha_5 x_1^3 + \alpha_6 x_1^2 x_2 + \alpha_7 x_1 x_2^2 + \alpha_8 x_2^3\end{bmatrix} \tag{2.104}$$

Substituting (2.104) into (2.8) yields

$$\begin{bmatrix}g_{11}\\ g_{12}\end{bmatrix} + \begin{bmatrix}\dfrac{\partial h_{11}}{\partial y_1} & \dfrac{\partial h_{11}}{\partial y_2}\\[2ex] \dfrac{\partial h_{12}}{\partial y_1} & \dfrac{\partial h_{12}}{\partial y_2}\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix} - \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}h_{11}\\ h_{12}\end{bmatrix}$$
$$= \begin{bmatrix}\alpha_1 y_1^3 + \alpha_2 y_1^2 y_2 + \alpha_3 y_1 y_2^2 + \alpha_4 y_2^3\\ \alpha_5 y_1^3 + \alpha_6 y_1^2 y_2 + \alpha_7 y_1 y_2^2 + \alpha_8 y_2^3\end{bmatrix} \tag{2.105}$$

The right-hand side of (2.105) suggests seeking the h_{1m} and g_{1m} in the form (2.28)–(2.31). Thus, substituting (2.28)–(2.31) into (2.105) yields

$$\begin{aligned} &\Lambda_1 y_1^3 + \Lambda_2 y_1^2 y_2 + \Lambda_3 y_1 y_2^2 + \Lambda_4 y_2^3 + y_2 \left(3\Gamma_1 y_1^2 + 2\Gamma_2 y_1 y_2 + \Gamma_3 y_2^2\right) \\ &\quad - \left(\Gamma_5 y_1^3 + \Gamma_6 y_1^2 y_2 + \Gamma_7 y_1 y_2^2 + \Gamma_8 y_2^3\right) \\ &\quad = \alpha_1 y_1^3 + \alpha_2 y_1^2 y_2 + \alpha_3 y_1 y_2^2 + \alpha_4 y_2^3 \end{aligned} \tag{2.106}$$

$$\begin{aligned} &\Lambda_5 y_1^3 + \Lambda_6 y_1^2 y_2 + \Lambda_7 y_1 y_2^2 + \Lambda_8 y_2^3 + y_2 \left(3\Gamma_5 y_1^2 + 2\Gamma_6 y_1 y_2 + \Gamma_7 y_2^2\right) \\ &\quad = \alpha_5 y_1^3 + \alpha_6 y_1^2 y_2 + \alpha_7 y_1 y_2^2 + \alpha_8 y_2^3 \end{aligned} \tag{2.107}$$

Equating the coefficients of $y_1^3, y_1^2 y_2, y_1 y_2^2$, and y_2^3 on both sides of (2.106) and (2.107), we obtain (2.18), where $\boldsymbol{\Gamma}$, $\boldsymbol{\alpha}$, and $\boldsymbol{\Lambda}$ are column vectors having the components Γ_m, α_m, and Λ_m, respectively, and

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{2.108}$$

Again, because B is a singular matrix, the system of (2.18) has a solution if and only if $\boldsymbol{\alpha} - \boldsymbol{\Lambda}$ is orthogonal to every solution of the adjoint homogeneous problem (2.75) and (2.108) whose nontrivial solutions in this case are

$$(0,0,0,0,1,0,0,0)^{\mathrm{T}} \quad \text{and} \quad (3,0,0,0,0,1,0,0)^{\mathrm{T}}$$

Consequently, (2.18) has a solution if and only if

$$\alpha_5 - \Lambda_5 = 0 \quad \text{and} \quad 3(\alpha_1 - \Lambda_1) + \alpha_6 - \Lambda_6 = 0$$

or

$$\Lambda_5 = \alpha_5 \tag{2.109}$$

$$3\Lambda_1 + \Lambda_6 = 3\alpha_1 + \alpha_6 \tag{2.110}$$

Then, one can solve (2.18) for $\boldsymbol{\Gamma}$ for all values of the α_m, and hence $\Lambda_2, \Lambda_3, \Lambda_4, \Lambda_7$, and Λ_8 can be set equal to zero. Consequently, the normal form of (2.104) is

$$\dot{y}_1 = y_2 + \epsilon \Lambda_1 y_1^3 \tag{2.111}$$

$$\dot{y}_2 = \epsilon \left(\alpha_5 y_1^3 + \Lambda_6 y_1^2 y_2\right) \tag{2.112}$$

where Λ_1 and Λ_6 satisfy the condition (2.110), and hence this normal form is not unique.

Again, because the linear part of the system under consideration cannot be diagonalized, the resonance terms are not spanned by the null space of B and hence this condition cannot be used to uniquely specify the normal form. In fact, using this condition would lead to the contradictory results $\alpha_5 = 0$ and $3\alpha_1 + \alpha_6 = 0$.

2.7
A Three-Dimensional System with Zero and Two Purely Imaginary Eigenvalues

We start with a system having quadratic nonlinearities of the form

$$\dot{x}_1 = x_2 + \epsilon\left(\delta_1 x_1^2 + \delta_2 x_1 x_2 + \delta_3 x_2^2 + \delta_4 w^2 + \delta_5 x_1 w + \delta_6 x_2 w\right) \quad (2.113)$$

$$\dot{x}_2 = -x_1 + \epsilon\left(\delta_7 x_1^2 + \delta_8 x_1 x_2 + \delta_9 x_2^2 + \delta_{10} w^2 + \delta_{11} x_1 w + \delta_{12} x_2 w\right) \quad (2.114)$$

$$\dot{w} = \epsilon\left(\delta_{13} x_1^2 + \delta_{14} x_1 x_2 + \delta_{15} x_2^2 + \delta_{16} w^2 + \delta_{17} x_1 w + \delta_{18} x_2 w\right) \quad (2.115)$$

so that

$$J = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

As discussed in Section 2.5, it is convenient to lump the coordinates x_1 and x_2 corresponding to the purely imaginary eigenvalues $\pm i$ into a single complex variable. To this end, we let

$$x_1 = \zeta + \bar{\zeta} \quad \text{and} \quad x_2 = i\left(\zeta - \bar{\zeta}\right) \quad (2.116)$$

and transform (2.113)–(2.115) into

$$\begin{aligned} \dot{\zeta} = i\zeta + \tfrac{1}{2}\epsilon\Big[&(\delta_1 - i\delta_7)\left(\zeta + \bar{\zeta}\right)^2 + (i\delta_2 + \delta_8)\left(\zeta^2 - \bar{\zeta}^2\right) \\ &- (\delta_3 - i\delta_9)\left(\zeta - \bar{\zeta}\right)^2 + (\delta_4 - i\delta_{10})\, w^2 + (\delta_5 - i\delta_{11})\, w\left(\zeta + \bar{\zeta}\right) \\ &+ (i\delta_6 + \delta_{12})\, w\left(\zeta - \bar{\zeta}\right)\Big] \end{aligned} \quad (2.117)$$

$$\begin{aligned} \dot{w} = \epsilon\Big[&\delta_{13}\left(\zeta + \bar{\zeta}\right)^2 + i\delta_{14}\left(\zeta^2 - \bar{\zeta}\right)^2 - \delta_{15}\left(\zeta - \bar{\zeta}\right)^2 + \delta_{16} w^2 \\ &+ \delta_{17} w\left(\zeta + \bar{\zeta}\right) + i\delta_{18} w\left(\zeta - \bar{\zeta}\right)\Big] \end{aligned} \quad (2.118)$$

The transformation (2.116) was chosen so that the unperturbed parts of (2.113) and (2.114) assume the simple form $\dot{\zeta} = i\zeta$.

To $O(1)$, (2.117) and (2.118) are

$$\dot{\zeta} = i\zeta \quad \text{and} \quad \dot{w} = 0 \quad (2.119)$$

Hence, $\zeta \propto e^{it}$ and w is independent of t. Consequently, only the term $w\zeta$ is a resonance term in (2.117) and the terms $\zeta\bar{\zeta}$ and w^2 are resonance terms in (2.118). Therefore, keeping only the resonance terms in (2.117) and (2.118), we obtain to first order the normal form

$$\dot{\zeta} = i\zeta + \tfrac{1}{2}\epsilon\left(\delta_5 + \delta_{12} + i\delta_6 - i\delta_{11}\right) w\zeta \tag{2.120}$$

$$\dot{w} = \epsilon\left[2\left(\delta_{13} + \delta_{15}\right)\zeta\bar{\zeta} + \delta_{16}w^2\right] \tag{2.121}$$

Expressing ζ in the polar form $1/2re^{i\beta}$, we rewrite (2.120) and (2.121) in the following alternate normal form:

$$\dot{r} = \tfrac{1}{2}\epsilon\left(\delta_5 + \delta_{12}\right) rw \tag{2.122}$$

$$\dot{w} = \tfrac{1}{2}\epsilon\left(\delta_{13} + \delta_{15}\right) r^2 + \epsilon\delta_{16}w^2 \tag{2.123}$$

$$\dot{\beta} = 1 + \tfrac{1}{2}\epsilon\left(\delta_6 - \delta_{11}\right) w \tag{2.124}$$

Next, we consider a system having cubic nonlinearities of the form

$$\begin{aligned}\dot{x}_1 = x_2 + \epsilon\big(&\alpha_1 x_1^3 + \alpha_2 x_1^2 x_2 + \alpha_3 x_1 x_2^2 + \alpha_4 x_2^3 + \alpha_5 x_1^2 w\\ &+\alpha_6 x_1 w^2 + \alpha_7 x_2^2 w + \alpha_8 x_2 w^2 + \alpha_9 x_1 x_2 w + \alpha_{10} w^3\big)\end{aligned} \tag{2.125}$$

$$\begin{aligned}\dot{x}_1 = -x_1 + \epsilon\big(&\alpha_{11} x_1^3 + \alpha_{12} x_1^2 x_2 + \alpha_{13} x_1 x_2^2 + \alpha_{14} x_2^3 + \alpha_{15} x_1^2 w\\ &+\alpha_{16} x_1 w^2 + \alpha_{17} x_2^2 w + \alpha_{18} x_2 w^2 + \alpha_{19} x_1 x_2 w + \alpha_{20} w^3\big)\end{aligned} \tag{2.126}$$

$$\begin{aligned}\dot{w} = \epsilon\big(&\alpha_{21} x_1^3 + \alpha_{22} x_1^2 x_2 + \alpha_{23} x_1 x_2^2 + \alpha_{24} x_2^3 + \alpha_{25} x_1^2 w + \alpha_{26} x_1 w^2\\ &+\alpha_{27} x_2^2 w + \alpha_{28} x_2 w^2 + \alpha_{29} x_1 x_2 w + \alpha_{30} w^3\big)\end{aligned} \tag{2.127}$$

Again, we use the transformation (2.116) to lump the coordinates x_1 and x_2 into a single complex variable and rewrite (2.125)–(2.127) as

$$\begin{aligned}\dot{\zeta} = i\zeta + \tfrac{1}{2}\epsilon\Big[&\left(\alpha_1 - i\alpha_{11}\right)\left(\zeta + \bar{\zeta}\right)^3 + \left(i\alpha_2 + \alpha_{12}\right)\left(\zeta + \bar{\zeta}\right)^2\left(\zeta - \bar{\zeta}\right)\\ &- \left(\alpha_3 - i\alpha_{13}\right)\left(\zeta + \bar{\zeta}\right)\left(\zeta - \bar{\zeta}\right)^2 - \left(i\alpha_4 + \alpha_{14}\right)\left(\zeta - \bar{\zeta}\right)^3\\ &+ \left(\alpha_5 - i\alpha_{15}\right) w\left(\zeta + \bar{\zeta}\right)^2 + \left(\alpha_6 - i\alpha_{16}\right) w^2\left(\zeta + \bar{\zeta}\right)\\ &- \left(\alpha_7 - i\alpha_{17}\right) w\left(\zeta - \bar{\zeta}\right)^2 + \left(i\alpha_8 + \alpha_{18}\right) w^2\left(\zeta - \bar{\zeta}\right)\\ &+ \left(i\alpha_9 + \alpha_{19}\right) w\left(\zeta^2 - \bar{\zeta}^2\right) + \left(\alpha_{10} - i\alpha_{20}\right) w^3\Big]\end{aligned} \tag{2.128}$$

$$\begin{aligned}\dot{w} = \epsilon\Big[&\alpha_{21}\left(\zeta + \bar{\zeta}\right)^3 + i\alpha_{22}\left(\zeta + \bar{\zeta}\right)^2\left(\zeta - \bar{\zeta}\right) - \alpha_{23}\left(\zeta + \bar{\zeta}\right)\left(\zeta - \bar{\zeta}\right)^2\\ &- i\alpha_{24}\left(\zeta - \bar{\zeta}\right)^3 + \alpha_{25} w\left(\zeta + \bar{\zeta}\right)^2 + \alpha_{26} w^2\left(\zeta + \bar{\zeta}\right) - \alpha_{27} w\left(\zeta - \bar{\zeta}\right)^2\\ &+i\alpha_{28} w^2\left(\zeta - \bar{\zeta}\right) + i\alpha_{29} w\left(\zeta^2 - \bar{\zeta}^2\right) + \alpha_{30} w^3\Big]\end{aligned} \tag{2.129}$$

Because, to $O(1)$, ζ oscillates with the frequency 1 and w has a zero frequency, only the terms $\zeta^2\bar{\zeta}$ and $w^2\zeta$ are resonance terms in (2.128) and only the terms $\zeta\bar{\zeta}w$ and w^3 are resonance terms in (2.129). Consequently, keeping only the resonance terms in (2.128) and (2.129), we obtain the normal form

$$\dot{\zeta} = i\zeta + \tfrac{1}{2}\epsilon\,[3\alpha_1 + \alpha_3 + \alpha_{12} + 3\alpha_{14} + i\,(\alpha_2 + 3\alpha_4 - 3\alpha_{11} - \alpha_{13})]\,\zeta^2\bar{\zeta} + \tfrac{1}{2}\epsilon\,[\alpha_6 + \alpha_{18} + i\,(\alpha_8 - \alpha_{16})]\,w^2\zeta \tag{2.130}$$

$$\dot{w} = 2\epsilon\,(\alpha_{25} + \alpha_{27})\,\zeta\bar{\zeta}w + \epsilon\alpha_{30}w^3 \tag{2.131}$$

Expressing ζ in the polar form $1/2re^{i\beta}$, we rewrite (2.130) and (2.131) as

$$\dot{r} = \tfrac{1}{8}\epsilon\,(3\alpha_1 + \alpha_3 + \alpha_{12} + 3\alpha_{14})\,r^3 + \tfrac{1}{2}\epsilon\,(\alpha_6 + \alpha_{18})\,w^2 r \tag{2.132}$$

$$\dot{w} = \tfrac{1}{2}\epsilon\,(\alpha_{25} + \alpha_{27})\,r^2 w + \epsilon\alpha_{30}w^3 \tag{2.133}$$

$$\dot{\beta} = 1 + \tfrac{1}{8}\epsilon\,(\alpha_2 + 3\alpha_4 - 3\alpha_{11} - \alpha_{13})\,r^2 + \tfrac{1}{2}\epsilon\,(\alpha_8 - \alpha_{16})\,w^2 \tag{2.134}$$

2.8
The Mathieu Equation

In Section 7.1, we use the method of normal forms to determine approximations to the solutions of the Mathieu equation

$$\ddot{u} + \delta u + 2\epsilon u \cos \Omega t = 0 \tag{2.135}$$

when $\delta \approx 1$ or 4. In this section, we consider the case $\delta \approx 0$, for which the two eigenvalues of the unperturbed equation are approximately zero. To accomplish this, we let $\delta = \epsilon\delta_1 + \epsilon^2\delta_2 + \cdots$ so that the eigenvalues of the unperturbed equation are exactly zero. Then, we cast (2.135) as a system of three first-order equations by using the transformation

$$x_1 = u\,, \quad x_2 = \dot{u}\,, \quad \text{and} \quad z = e^{i\Omega t} \tag{2.136}$$

Hence, (2.135) can be rewritten as

$$\dot{x}_1 = x_2 \tag{2.137}$$

$$\dot{x}_2 = -\left(\epsilon\delta_1 + \epsilon^2\delta_2\right)x_1 - \epsilon x_1\,(z + \bar{z}) \tag{2.138}$$

$$\dot{z} = i\Omega z \tag{2.139}$$

To simplify (2.137) and (2.138), we let

$$x_1 = y_1 + \epsilon h_{11}(y_1, y_2, z, \bar{z}) + \epsilon^2 h_{12}(y_1, y_2, z, \bar{z}) + \cdots \tag{2.140}$$

$$x_2 = y_2 + \epsilon h_{21}(y_1, y_2, z, \bar{z}) + \epsilon^2 h_{22}(y_1, y_2, z, \bar{z}) + \cdots \tag{2.141}$$

$$\dot{y}_1 = y_2 + \epsilon g_{11}(y_1, y_2, z, \bar{z}) + \epsilon^2 g_{12}(y_1, y_2, z, \bar{z}) + \cdots \tag{2.142}$$

$$\dot{y}_2 = \epsilon g_{21}(y_1, y_2, z, \bar{z}) + \epsilon^2 g_{22}(y_1, y_2, z, \bar{z}) + \cdots \tag{2.143}$$

where the g_{mn} contain only resonance terms. Substituting (2.140)–(2.143) into (2.137) and (2.138) and equating coefficients of like powers of ϵ, we obtain

$$g_{11} + \frac{\partial h_{11}}{\partial y_1} y_2 + i\Omega \frac{\partial h_{11}}{\partial z} z - i\Omega \frac{\partial h_{11}}{\partial \bar{z}} \bar{z} - h_{21} = 0 \tag{2.144}$$

$$g_{21} + \frac{\partial h_{21}}{\partial y_1} y_2 + i\Omega \frac{\partial h_{21}}{\partial z} z - i\Omega \frac{\partial h_{21}}{\partial \bar{z}} \bar{z} = -\delta_1 y_1 - y_1(z + \bar{z}) \tag{2.145}$$

$$g_{12} + \frac{\partial h_{12}}{\partial y_1} y_2 + i\Omega \frac{\partial h_{12}}{\partial z} z - i\Omega \frac{\partial h_{12}}{\partial \bar{z}} \bar{z} - h_{22} = -g_{11} \frac{\partial h_{11}}{\partial y_1} - g_{21} \frac{\partial h_{11}}{\partial y_2} \tag{2.146}$$

$$g_{22} + \frac{\partial h_{22}}{\partial y_1} y_2 + i\Omega \frac{\partial h_{22}}{\partial z} z - i\Omega \frac{\partial h_{22}}{\partial \bar{z}} \bar{z} = -\delta_1 h_{11} - \delta_2 y_1 - g_{11} \frac{\partial h_{21}}{\partial y_1} - g_{21} \frac{\partial h_{21}}{\partial y_2} - h_{11}(z + \bar{z}) \tag{2.147}$$

Because the frequency of z is Ω and, to first order, the frequencies of x_1 and x_2 are zero, the perturbation term $y_1(z + \bar{z})$ in (2.145) does not produce any resonances, and hence we put $\delta_1 = 0$ and then $g_{11} = g_{21} = 0$. Consequently, we seek the solution of (2.144) and (2.145) in the form

$$h_{11} = \Gamma_1 y_1 z + \Gamma_2 y_1 \bar{z} + \Gamma_3 y_2 z + \Gamma_4 y_2 \bar{z} \tag{2.148}$$

$$h_{21} = \Gamma_5 y_1 z + \Gamma_6 y_1 \bar{z} + \Gamma_7 y_2 z + \Gamma_8 y_2 \bar{z} \tag{2.149}$$

Equating the coefficients of $y_m z$ and $y_m \bar{z}$ on both sides, we obtain

$$\begin{aligned} &i\Omega\Gamma_1 - \Gamma_5 = 0\,, \quad i\Omega\Gamma_2 + \Gamma_6 = 0\,, \\ &\Gamma_2 - i\Omega\Gamma_4 - \Gamma_8 = 0\,, \quad \Gamma_5 + i\Omega\Gamma_7 = 0\,, \\ &i\Omega\Gamma_5 = -1\,, \quad \Gamma_1 + i\Omega\Gamma_3 - \Gamma_7 = 0\,, \\ &i\Omega\Gamma_6 = 1\,, \quad \Gamma_6 - i\Omega\Gamma_8 = 0 \end{aligned}$$

Hence,

$$(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4) = \frac{1}{\Omega^2}\left(1, 1, \frac{2i}{\Omega}, -\frac{2i}{\Omega}\right) \tag{2.150}$$

$$(\Gamma_5, \Gamma_6, \Gamma_7, \Gamma_8) = \frac{i}{\Omega}\left(1, -1, \frac{i}{\Omega}, \frac{i}{\Omega}\right) \tag{2.151}$$

Using (2.150) and (2.151) and the fact that $g_{11} = g_{21} = 0$, we rewrite (2.146) and (2.147) as

$$g_{12} + \frac{\partial h_{12}}{\partial y_1} y_2 + i\Omega \frac{\partial h_{12}}{\partial z} z - i\Omega \frac{\partial h_{12}}{\partial z} \bar{z} - h_{22} = 0 \tag{2.152}$$

$$g_{22} + \frac{\partial h_{22}}{\partial y_1} y_2 + i\Omega \frac{\partial h_{22}}{\partial z} z - i\Omega \frac{\partial h_{22}}{\partial \bar{z}} \bar{z}$$
$$= -\delta_2 y_1 - \frac{1}{\Omega^2} \left[y_1 (z + \bar{z}) + \frac{2i}{\Omega} y_2 (z - \bar{z}) \right] (z + \bar{z}) \tag{2.153}$$

Because the nonhomogeneous part in (2.152) is zero, there are no resonance terms and hence $g_{12} = 0$. Inspection of the right-hand side of (2.153) shows that $\delta_2 y_1$ and $y_1 z \bar{z}$ are resonance terms and hence

$$g_{22} = -\delta_2 y_1 - \frac{2}{\Omega^2} z \bar{z} y_1 \tag{2.154}$$

Consequently, to second order, the normal form of (2.137) and (2.138) is

$$\dot{y}_1 = y_2 \tag{2.155}$$

$$\dot{y}_2 = -\epsilon^2 \left(\delta_2 + \frac{2}{\Omega^2} \right) y_1 \tag{2.156}$$

which, upon elimination of y_2, yields

$$\ddot{y}_1 = -\epsilon^2 \left(\delta_2 + \frac{2}{\Omega^2} \right) y_1 \tag{2.157}$$

Next, we use the method of multiple scales to determine a second-order uniform expansion of (2.135) when $\delta = \epsilon \delta_2 + \epsilon^2 \delta_2$. To this end, we seek the expansion of u in the form

$$u = u_0 (T_0, T_1, T_2) + \epsilon u_1 (T_0, T_1, T_2) + \epsilon^2 u_2 (T_0, T_1, T_2) + \cdots \tag{2.158}$$

where $T_n = \epsilon^n t$. In terms of the T_n,

$$\frac{d}{dt} = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \cdots, \quad D_n = \frac{\partial}{\partial T_n} \tag{2.159}$$

Substituting (2.158) and (2.159) into (2.135) and equating coefficients of like powers of ϵ, we have

$$D_0^2 u_0 = 0 \tag{2.160}$$

$$D_0^2 u_1 = -2 D_0 D_1 u_0 - 2 u_1 \cos \Omega T_0 - \delta_1 u_0 \tag{2.161}$$

$$D_0^2 u_2 = -2 D_0 D_2 u_0 - D_1^2 u_0 - 2 D_0 D_1 u_1 - \delta_2 u_0 - 2 u_1 \cos \Omega T_0 \tag{2.162}$$

The solution of (2.160) can be expressed as

$$u_0 = A(T_1, T_2) \tag{2.163}$$

Then, (2.161) becomes

$$D_0^2 u_1 = -2A \cos \Omega\, T_0 - \delta_1 A$$

Hence, u_1 will have a secular term unless $\delta_1 = 0$. Then,

$$u_1 = \frac{2A}{\Omega^2} \cos \Omega\, T_0 \tag{2.164}$$

Substituting (2.163) and (2.164) into (2.162) yields

$$D_0^2 u_2 = -D_1^2 A - \frac{2A}{\Omega^2} - \delta_2 A + \text{NST}\,. \tag{2.165}$$

Eliminating the terms that produce secular terms from (2.165) yields

$$D_1^2 A = -\left(\delta_2 + \frac{2}{\Omega^2}\right) A \tag{2.166}$$

which is in full agreement with (2.157), yet it is obtained with much less algebra.

2.9 Exercises

2.9.1 Determine the normal form of

$$\dot{u} = Au$$

where

a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$,

b) $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$,

c) $A = \begin{bmatrix} 2 & -3 \\ 2 & 1 \end{bmatrix}$,

d) $A = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$,

e) $A = \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix}$,

f) $A = \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix}$,

g) $A = \begin{bmatrix} 1 & -5 \\ 2 & -1 \end{bmatrix}$.

2.9.2 Determine the normal form of

$$\dot{u} = Au$$

where

a) $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix}$,

b) $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ 3 & -6 & 6 \end{bmatrix}$,

c) $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$,

d) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$,

e) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{bmatrix}$,

f) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 0 \end{bmatrix}$.

2.9.3 Determine the normal forms of

a) $\dot{u}_1 = 3u_1 + u_2 + \alpha_1 u_1^2$

$\dot{u}_2 = -u_1 + u_2 + \alpha_2 u_2^2$,

b) $\dot{u}_1 = -u_1 + u_2 + \alpha_1 u_1^2$

$\dot{u}_2 = -u_1 - u_2 + \alpha_2 u_2^2$,

c) $\dot{u}_1 = -u_1 + u_2 + \alpha_1 u_1^2$

$\dot{u}_2 = u_1 - u_2 + \alpha_2 u_2^2$,

d) $\dot{u}_1 = -u_1 + 2u_2 + \alpha_1 u_1 u_2$

$\dot{u}_2 = 2u_1 - u_2 + \alpha_2 u_1^2$.

2.9.4 Determine the normal forms of

a) $\dot{u}_1 = -u_1 + 2u_2 + \alpha_1 u_1^3 + \alpha_2 u_1^2 u_2$
$\dot{u}_2 = 2u_1 - u_2 + \alpha_3 u_1^3 + \alpha_4 u_1 u_2$,
b) $\dot{u}_1 = -u_1 + u_2 + \alpha_1 u_1^3 + \alpha_2 u_1^2 u_2$
$\dot{u}_2 = -u_1 - u_2 + \alpha_3 u_1^3 + \alpha_4 u_1 u_2^2$,
c) $\dot{u}_1 = -u_1 + u_2 + \alpha_1 u_1^3 + \alpha_2 u_1^2 u_2$
$\dot{u}_2 = u_1 - u_2 + \alpha_3 u_1^3 + \alpha_4 u_1 u_2^2$,
d) $\dot{u}_1 = 3u_1 + u_2 + \alpha_1 u_1^3 + \alpha_2 u_1^2 u_2$
$\dot{u}_2 = u_1 - u_2 + \alpha_3 u_1^3 + \alpha_4 u_1 u_2^2$.

2.9.5 Determine the normal forms of

a) $\dot{u} = Au + \begin{bmatrix} \alpha_1 u_1^2 \\ \alpha_2 u_1 u_3 \\ \alpha_3 u_2^2 \end{bmatrix}$

b) $\dot{u} = Au + \begin{bmatrix} \alpha_1 u_1^3 \\ \alpha_2 u_1^2 u_2 + \alpha_3 u_1 u_3^2 \\ \alpha_4 u_3^3 \end{bmatrix}$

where

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 4 & -3 & 2 \\ 8 & -4 & 3 \end{bmatrix} .$$

2.9.6 Determine the normal forms of

a) $\dot{u} = Au + \begin{bmatrix} \alpha_1 u_1^2 \\ \alpha_2 u_1 u_3 \\ \alpha_3 u_2^2 \end{bmatrix}$

b) $\dot{u} = Au + \begin{bmatrix} \alpha_1 u_1^3 \\ \alpha_2 u_1^2 u_2 + \alpha_3 u_1 u_2^2 \\ \alpha_4 u_3^3 \end{bmatrix}$

where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{bmatrix} .$$

2.9.7 Determine the normal form of

a) $\dot{u}_1 = -u_1 + 2u_2 + \alpha_1 u_1 u_3 + \alpha_2 u_1^3$,

b) $\dot{u}_2 = -u_2 - 2u_1 + \alpha_3 u_2 u_3 + \alpha_4 u_2^3$,
c) $\dot{u}_3 = -u_3 + \alpha_5 u_1^2 + \alpha_6 u_1 u_2 + \alpha_7 u_2^2$.

2.9.8 Determine the normal form of

$$\dot{u} = \begin{bmatrix} \frac{3}{2} & 0 & \frac{1}{2} \\ -\frac{9}{2} & 0 & -\frac{3}{2} \\ -\frac{23}{2} & 2 & -\frac{9}{2} \end{bmatrix} u + \begin{bmatrix} \alpha_1 u_1 u_2^2 \\ \alpha_2 u_2 u_3^2 \\ \alpha_3 u_1^3 \end{bmatrix}$$

2.9.9 Determine the second-order normal form of

$$\dot{x}_1 = -x_2 + \alpha_1 x_1^2 + \alpha_2 x_1 x_2$$
$$\dot{x}_2 = x_1 + \alpha_3 x_2^2$$

2.9.10 Determine the normal form of

$$\dot{u} = \begin{bmatrix} \frac{3}{2} & 0 & \frac{1}{2} \\ -\frac{5}{2} & -6 & \frac{5}{2} \\ -\frac{15}{2} & -10 & \frac{7}{2} \end{bmatrix} u + \begin{bmatrix} \alpha_1 u_1^2 u_2 \\ \alpha_2 u_2^3 \\ \alpha_3 u_1 u_3^2 \end{bmatrix}$$

3
Maps

In this chapter, we construct normal forms of smooth maps depending on a scalar control parameter μ near their fixed or equilibrium points. Specifically, we consider the following map:

$$\boldsymbol{x}_{k+1} = \boldsymbol{F}(\boldsymbol{x}_k; \mu)$$

where $\boldsymbol{x} \in U \subset \mathcal{R}^n$, $\boldsymbol{F} \in U \subset \mathcal{R}^n$, $\mu \in V \subset \mathcal{R}$, and $\boldsymbol{x}_k$ and $\boldsymbol{x}_{k+1}$ represent the states of the system at the discrete times t_k and t_{k+1}, respectively. The fixed points of this map are solutions of the algebraic system of equations

$$\boldsymbol{F}(\boldsymbol{x}; \mu) = \boldsymbol{x}$$

In Section 3.1, we consider linear maps; in Section 3.2, we consider nonlinear maps; in Section 3.3, we discuss center-manifold reduction; and in Section 3.4, we consider bifurcation of smooth maps.

3.1
Linear Maps

We consider the linear map

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k \tag{3.1}$$

where A is an $n \times n$ constant matrix. In this case, the trivial solution $\boldsymbol{x}^* = \boldsymbol{0}$ is a fixed point of this linear map. We denote the eigenvalues of A by ρ_i, $i = 1, 2, \ldots, n$, and the corresponding eigenvectors (generalized eigenvectors) by $\boldsymbol{p}_i$, $i = 1, 2, \ldots, n$. The eigenvalues are the roots of the characteristic equation

$$\det(A - \rho I) = 0 \tag{3.2}$$

The eigenvector $\boldsymbol{p}_i$ corresponding to a distinct eigenvalue ρ_i is given by

$$A\boldsymbol{p}_i = \rho_i \boldsymbol{p}_i \tag{3.3}$$

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and the generalized eigenvectors corresponding to an eigenvalue ρ_m with multiplicity n_m are the nontrivial solutions of

$$(A - \rho_m \boldsymbol{I})\boldsymbol{p} = \boldsymbol{0}\,, \quad (A - \rho_m \boldsymbol{I})^2 \boldsymbol{p} = \boldsymbol{0}\,, \quad \cdots, \quad (A - \rho_m \boldsymbol{I})^{n_m} \boldsymbol{p} = \boldsymbol{0} \tag{3.4}$$

Introducing the transformation

$$\boldsymbol{x} = P\boldsymbol{y} \tag{3.5}$$

where the matrix $P = [\boldsymbol{p}_1 \boldsymbol{p}_2 \cdots \boldsymbol{p}_n]$, into (3.1) yields

$$P\boldsymbol{y}_{k+1} = AP\boldsymbol{y}_k \tag{3.6}$$

Multiplying (3.6) from the left with the inverse P^{-1} of P, we obtain from (3.6) that the normal form of (3.1) is

$$\boldsymbol{y}_{k+1} = J\boldsymbol{y}_k \tag{3.7}$$

where $J = P^{-1}AP$ is called the Jordan canonical form of A. Next, we discuss two cases: maps with distinct and nondistinct eigenvalues.

3.1.1
Case of Distinct Eigenvalues

If the eigenvalues of A are distinct, then J is a diagonal matrix D with entries ρ_i, $i = 1, 2, \ldots, n$; that is,

$$D = \begin{bmatrix} \rho_1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \rho_2 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \rho_n \end{bmatrix} \tag{3.8}$$

Then, (3.7) can be rewritten as

$$y_{k+1}^{(m)} = \rho_m y_k^{(m)}\,, \quad m = 1, 2, \ldots, n \tag{3.9}$$

where $y^{(m)}$ is the mth component of $\boldsymbol{y}$. Consequently,

$$y_k^{(m)} = \rho_m^k y_0^{(m)}\,, \quad m = 1, 2, \ldots, n \tag{3.10}$$

It follows from (3.10) that $y^{(m)} \to 0$ as $k \to \infty$ when ρ_m is inside the unit circle in the complex plane and $y^{(m)} \to \infty$ as $k \to \infty$ when ρ_m is outside the unit circle in the complex plane.

Example 3.1

We consider a map with the matrix

$$A = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$$

The eigenvalues of this matrix are 1/2 and 3/2 and their corresponding eigenvectors are the columns of the following matrix P:

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Then,

$$P^{-1}AP = D = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix}$$

and the normal form of the corresponding map is

$$y_{k+1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix} y_k \tag{3.11}$$

Example 3.2

We consider a map with the matrix

$$A = \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}$$

The eigenvalues of this matrix are $1 + 2i$ and $1 - 2i$ and their corresponding eigenvectors are the columns of the following matrix P:

$$P = \begin{bmatrix} 2i & -2i \\ 1 & 1 \end{bmatrix}$$

In contrast with the preceding example, the eigenvalues and eigenvectors of this matrix are complex-valued rather than real-valued. Then,

$$P^{-1}AP = D = \begin{bmatrix} 1+2i & 0 \\ 0 & 1-2i \end{bmatrix}$$

Hence, the normal form of the corresponding map can be expressed in complex-valued form as

$$y_{k+1} = \begin{bmatrix} 1+2i & 0 \\ 0 & 1-2i \end{bmatrix} y_k \tag{3.12}$$

Expressing $y^{(m)}$ and $1 + 2i$ in polar form as

$$y^{(1)} = \tfrac{1}{2} a e^{i\theta} , \quad y^{(2)} = \tfrac{1}{2} a e^{-i\theta} , \quad 1 + 2i = r e^{i\omega}$$

where $r = \sqrt{5}$ and $\omega = \tan^{-1}(2)$, we rewrite (3.12) as

$$a_{k+1} = r a_k , \quad \theta_{k+1} = \theta_k + \omega \tag{3.13}$$

Hence, starting from $a = a_0$ and $\theta = \theta_0$, we have

$$a_k = r^k a_0 , \quad \theta_k = \theta_0 + k\omega \tag{3.14}$$

3.1.2
Case of Repeated Eigenvalues

If the number of distinct eigenvalues of A is $k < n$, then J has the form

$$J = \begin{bmatrix} J_1 & \phi & \cdot & \cdot & \cdot & \phi \\ \phi & J_2 & \cdot & \cdot & \cdot & \phi \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \phi & \phi & \cdot & \cdot & \cdot & J_k \end{bmatrix} \tag{3.15}$$

where ϕ represents a matrix with zero entries and

$$J_m = \begin{bmatrix} \rho_m & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \rho_m & 1 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \rho_m & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \rho_m \end{bmatrix} \tag{3.16}$$

Example 3.3

We consider the following map with repeated roots:

$$A = \begin{bmatrix} a & \tfrac{1}{2}(a-b) \\ -\tfrac{1}{2}(a-b) & b \end{bmatrix}$$

where $b \neq a$. The eigenvalues of this matrix are $\rho = 1/2(a + b)$ with a multiplicity of two and the corresponding eigenvectors are given by

$$\left(A - \tfrac{1}{2}(a+b) I\right) \boldsymbol{p} = 0$$

or

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence, we have the single condition $p_1 + p_2 = 0$, which relates p_2 to p_1 or vice versa. Choosing $p_2 = 1$, we have $p_1 = -1$ and obtain the eigenvector

$$\boldsymbol{p}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Because A has only one linearly independent eigenvector, we need to calculate a generalized eigenvector $\boldsymbol{p}_2$. To this end, we have

$$\left(A - \tfrac{1}{2}(a+b)I\right) \boldsymbol{p}_2 = \boldsymbol{p}_1$$

which yields the single condition

$$(b-a)(p_1 + p_2) = 2$$

Choosing $p_1 = -1$, we have

$$p_2 = 1 + \frac{2}{b-a}$$

and hence the generalized eigenvector can be expressed as

$$\boldsymbol{p}_2 = \begin{bmatrix} -1 \\ 1 + \dfrac{2}{b-a} \end{bmatrix}$$

Then,

$$P^{-1}AP = J = \begin{bmatrix} \frac{1}{2}(a+b) & 1 \\ 0 & \frac{1}{2}(a+b) \end{bmatrix}$$

and (3.7) yields

$$y_{k+1}^{(1)} = \rho y_k^{(1)} + y_k^{(2)} \tag{3.17}$$

$$y_{k+1}^{(2)} = \rho y_k^{(2)} \tag{3.18}$$

Starting from $y^{(1)} = y_0^{(1)}$ and $y^{(2)} = y_0^{(2)}$, we obtain from (3.17) and (3.18) that

$$y_k^{(1)} = \rho^k y_0^{(1)} + k\rho^{k-1} y_0^{(2)} \tag{3.19}$$

$$y_k^{(2)} = \rho^k y_0^{(2)} \tag{3.20}$$

3.2 Nonlinear Maps

In this section, we construct normal forms of the nonlinear n-dimensional map

$$\boldsymbol{x}_{k+1} = \boldsymbol{F}(\boldsymbol{x}_k; \mu) \tag{3.21}$$

where μ is the control parameter. A fixed point $\boldsymbol{x}^*$ at $\mu = \mu_0$ of this map satisfies the condition

$$\boldsymbol{x}^* = \boldsymbol{F}^{(m)}(\boldsymbol{x}^*; \mu_0) \quad \text{for all} \quad m \in \mathcal{Z} \tag{3.22}$$

where $\mathcal{Z}$ is the set of all positive integers. We note that an orbit of a map initiated at a fixed point of the map is the fixed point itself. Moreover, the fixed points of a map are examples of invariant sets.

To simplify (i.e., construct the normal form of) (3.21) near a fixed point, we first shift the fixed point from $\boldsymbol{x}^*$ to the origin by letting $\boldsymbol{x} = \boldsymbol{x}^* + \boldsymbol{y}$ and obtain

$$\boldsymbol{y}_{k+1} + \boldsymbol{x}^* = \boldsymbol{F}(\boldsymbol{y}_k + \boldsymbol{x}^*; \mu_0) \tag{3.23}$$

Then, we expand $\boldsymbol{F}$ in a Taylor series around $\boldsymbol{x}^*$, use (3.22), and obtain

$$\boldsymbol{y}_{k+1} = A\boldsymbol{y}_k + \boldsymbol{F}_2(\boldsymbol{y}_k; \mu_0) + \boldsymbol{F}_3(\boldsymbol{y}_k; \mu_0) + \cdots \tag{3.24}$$

where $A = D_{\boldsymbol{x}}\hat{\boldsymbol{F}}(\boldsymbol{x}^*; \mu_0)$ and $\hat{\boldsymbol{F}}_r(\boldsymbol{y})$ is a vector-valued monomial of degree r; that is,

$$\left(y_1^{m_1} y_2^{m_2} \dots y_n^{m_n}\right) \boldsymbol{e}_i$$

where $m_1 + m_2 + \ldots + m_n = r$, $m_j \neq 0$, and $(\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n)$ are a basis of $\mathcal{R}^n$. Next, we introduce the linear transformation

$$\boldsymbol{y} = P\boldsymbol{w} \tag{3.25}$$

into (3.24), where the columns of P are the (generalized) eigenvectors of A, multiply the outcome from the left by P^{-1}, and obtain

$$\boldsymbol{w}_{k+1} = J\boldsymbol{w}_k + P^{-1}\hat{\boldsymbol{F}}_2\left(P\boldsymbol{w}_k; \mu_0\right) + P^{-1}\hat{\boldsymbol{F}}_3\left(P\boldsymbol{w}_k; \mu_0\right) + \cdots \tag{3.26}$$

where J is the Jordan canonical form of A. To simplify (3.26) according to the method of normal forms, one introduces successive near-identity analytic transformations to eliminate as many quadratic terms (i.e., $\hat{\boldsymbol{F}}_2$) as possible, then as many cubic terms (i.e., $\hat{\boldsymbol{F}}_3$) as possible, and so on. Next, we show how to reduce (3.26) to its normal form when the lowest nonlinear terms are of order r.

To simplify the notation, we rewrite (3.26) as

$$\boldsymbol{x}_{k+1} = J\boldsymbol{x}_k + \boldsymbol{F}_r\left(\boldsymbol{x}_k; \mu\right) \tag{3.27}$$

Then, we introduce the near-identity transformation

$$\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{h}(\boldsymbol{y}) \tag{3.28}$$

where the elements of $\boldsymbol{h}$ are monomials of order r, in (3.27), eliminate as many nonlinear terms as possible, and obtain the normal form

$$\boldsymbol{y}_{k+1} = J\boldsymbol{y}_k + \boldsymbol{g}(\boldsymbol{y}_k) \tag{3.29}$$

where the terms in $\boldsymbol{g}$ cannot be eliminated by any choice of $\boldsymbol{h}$. They are called resonance or near-resonance terms.

Substituting (3.28) into (3.27) yields

$$\boldsymbol{y}_{k+1} + \boldsymbol{h}(\boldsymbol{y}_{k+1}) = J\boldsymbol{y}_k + J\boldsymbol{h}(\boldsymbol{y}_k) + \boldsymbol{F}_r\left[\boldsymbol{y}_k + \boldsymbol{h}(\boldsymbol{y}_k)\right] \tag{3.30}$$

Substituting (3.29) into (3.30), we have

$$J\boldsymbol{y} + \boldsymbol{g}(\boldsymbol{y}) + \boldsymbol{h}\left[J\boldsymbol{y} + \boldsymbol{g}(\boldsymbol{y})\right] = J\boldsymbol{y} + J\boldsymbol{h}(\boldsymbol{y}) + \boldsymbol{F}_r\left[\boldsymbol{y} + \boldsymbol{h}(\boldsymbol{y})\right] \tag{3.31}$$

Expanding the terms in (3.31) in Taylor series and keeping terms of degree r, we obtain

$$\boldsymbol{g}(\boldsymbol{y}) + \boldsymbol{h}(J\boldsymbol{y}) - J\boldsymbol{h}(\boldsymbol{y}) = \boldsymbol{F}_r(\boldsymbol{y}) \tag{3.32}$$

which is called the *homological equation*. Next, we expand $\boldsymbol{F}_r(\boldsymbol{y})$, $\boldsymbol{h}(\boldsymbol{y})$, and $\boldsymbol{g}(\boldsymbol{y})$ as

$$\boldsymbol{F}_r(\boldsymbol{y}) = \sum_{i=1}^{n} a_{\boldsymbol{m}i}\boldsymbol{y}^{\boldsymbol{m}}\boldsymbol{e}_i$$

$$\boldsymbol{h}(\boldsymbol{y}) = \sum_{i=1}^{n} b_{\boldsymbol{m}i}\boldsymbol{y}^{\boldsymbol{m}}\boldsymbol{e}_i$$

$$\boldsymbol{g}(\boldsymbol{y}) = \sum_{i=1}^{n} c_{\boldsymbol{m}i}\boldsymbol{y}^{\boldsymbol{m}}\boldsymbol{e}_i$$

and obtain from (3.32) that

$$c_{\boldsymbol{m}i} + (\rho^{\boldsymbol{m}} - \rho_i)\, b_{\boldsymbol{m}i} = a_{\boldsymbol{m}i} \tag{3.33}$$

or

$$b_{\boldsymbol{m}i} = \frac{a_{\boldsymbol{m}i} - c_{\boldsymbol{m}i}}{\rho^{\boldsymbol{m}} - \rho_i} \quad \text{for} \quad i = 1, 2, \ldots, n \tag{3.34}$$

It follows from (3.34) that $b_{\boldsymbol{m}i}$ is singular or near-singular if

$$\rho^{\boldsymbol{m}} \approx \rho_i \tag{3.35}$$

The condition (3.35) is called *resonance or near-resonance of order r*. If $\rho^{\boldsymbol{m}}$ is away from ρ_i, we let $c_{\boldsymbol{m}i} = 0$ and obtain

$$b_{\boldsymbol{m}i} = \frac{a_{\boldsymbol{m}i}}{\rho^{\boldsymbol{m}} - \rho_i}$$

If $\rho^m \approx \rho_i$, we let

$$c_{mi} = a_{mi}$$

and hence b_{mi} is arbitrary.

Example 3.4

We consider the map

$$\boldsymbol{x}_{k+1} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} \boldsymbol{x}_k + \begin{bmatrix} a_1 x_{1k}^2 + a_2 x_{1k} x_{2k} + a_3 x_{2k}^2 \\ a_4 x_{1k}^2 + a_5 x_{1k} x_{2k} + a_6 x_{2k}^2 \end{bmatrix} \tag{3.36}$$

In this case, $r = 2$ and

$$\boldsymbol{F} = a_1 x_1^2 \boldsymbol{e}_1 + a_2 x_1 x_2 \boldsymbol{e}_1 + a_3 x_2^2 \boldsymbol{e}_1 + a_4 x_1^2 \boldsymbol{e}_2 + a_5 x_1 x_2 \boldsymbol{e}_2 + a_6 x_2^2 \boldsymbol{e}_2$$

$$\boldsymbol{h} = b_1 x_1^2 \boldsymbol{e}_1 + b_2 x_1 x_2 \boldsymbol{e}_1 + b_3 x_2^2 \boldsymbol{e}_1 + b_4 x_1^2 \boldsymbol{e}_2 + b_5 x_1 x_2 \boldsymbol{e}_2 + b_6 x_2^2 \boldsymbol{e}_2$$

$$\boldsymbol{g} = c_1 x_1^2 \boldsymbol{e}_1 + c_2 x_1 x_2 \boldsymbol{e}_1 + c_3 x_2^2 \boldsymbol{e}_1 + c_4 x_1^2 \boldsymbol{e}_2 + c_5 x_1 x_2 \boldsymbol{e}_2 + c_6 x_2^2 \boldsymbol{e}_2$$

Therefore,

$$b_1 = \frac{a_1 - c_1}{\rho_1^2 - \rho_1}, \quad b_2 = \frac{a_2 - c_2}{\rho_1 \rho_2 - \rho_1}, \quad b_3 = \frac{a_3 - c_3}{\rho_2^2 - \rho_1}$$
$$b_4 = \frac{a_4 - c_4}{\rho_1^2 - \rho_2}, \quad b_5 = \frac{a_5 - c_5}{\rho_1 \rho_2 - \rho_2}, \quad b_6 = \frac{a_6 - c_6}{\rho_2^2 - \rho_2}$$

One or more of the b_i are singular or near-singular (have small divisors) if (a) either ρ_1 or ρ_2 is zero or near zero, (b) either ρ_1 or ρ_2 is equal or nearly equal to unity, (c) ρ_2 is equal or nearly equal to ρ_1^2, and (d) ρ_1 is equal or nearly equal to ρ_2^2. These conditions are *resonance or near-resonance conditions of order two* and can be written in compact form as

$$\rho_1^{m_1} \rho_2^{m_2} \approx \rho_i \quad \text{for} \quad i = 1 \text{ and } 2 \tag{3.37}$$

where $m_1 + m_2 = 2$.

When none of the b_i is singular or near-singular (i.e., absence of resonances or near-resonances of order two), $\boldsymbol{g} = 0$ and the transformation (3.28) yields the linear form

$$\boldsymbol{y}_{k+1} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} \boldsymbol{y}_k \tag{3.38}$$

In the presence of any resonance or near resonance, (3.36) cannot be reduced to a linear form. That is, when the ρ_i are such that one or more of the b_i are singular or near-singular, then one or more of the b_j are arbitrary and $c_j = a_j$. For example, in the presence of only the resonance condition $\rho_1^2 \approx \rho_2$, b_4 is arbitrary, $c_4 = a_4$, and $a_4 x_1^2$ is a resonance term, which cannot be eliminated by a proper choice of b_4. Hence the normal form of the map would be

$$y_{k+1} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} y_k + \begin{bmatrix} 0 \\ a_4 y_{1k}^2 \end{bmatrix} \tag{3.39}$$

And when $\rho_1 \approx \rho_2^2$, $a_3 x_2^2$ is a resonance term, it cannot be eliminated by a proper choice of b_3, and the normal form would be

$$y_{k+1} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} y_k + \begin{bmatrix} a_3 y_2^2 \\ 0 \end{bmatrix} \tag{3.40}$$

When $\rho_1 \approx 1$, the normal form would be

$$y_{k+1} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} y_k + \begin{bmatrix} a_1 y_{1k}^2 \\ a_5 y_{1k} y_{2k} \end{bmatrix} \tag{3.41}$$

When $\rho_2 \approx 1$, the normal form would be

$$y_{k+1} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} y_k + \begin{bmatrix} a_2 y_{1k} y_{2k} \\ a_6 y_{2k}^2 \end{bmatrix} \tag{3.42}$$

When $\rho_1 = e^{i\alpha}$, where $\alpha \neq 0$, then $\rho_2 = e^{-i\alpha}$ and $y^{(2)}$ must be the complex conjugate of $y^{(1)}$. Then, the resonance conditions (3.37) become

$$\rho_1^3 = 1$$

Hence,

$$\rho_1 = e^{\frac{2}{3} i\pi} \quad \text{or} \quad \rho_1 = e^{-\frac{2}{3} i\pi}$$

The normal form of the map when $\rho_1 = e^{\frac{2}{3} i\pi}$ would be

$$\xi_{k+1} = e^{\frac{2}{3} i\pi} \xi_k + a_3 \bar{\xi}_k^2 \tag{3.43}$$

where $\xi = y^{(1)}$. The normal form of the map when $\rho_1 = e^{-\frac{2}{3} i\pi}$ would be the complex conjugate of that given by (3.43).

Example 3.5

We consider the map

$$x_{k+1} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} x_k + \begin{bmatrix} a_1 x_{1k}^3 + a_2 x_{1k}^2 x_{2k} + a_3 x_{1k} x_{2k}^2 + a_4 x_{2k}^3 \\ a_5 x_{1k}^3 + a_6 x_{1k}^2 x_{2k} + a_7 x_{1k} x_{2k}^2 + a_8 x_{2k}^3 \end{bmatrix} \tag{3.44}$$

In this case, $r = 3$ and

$$\begin{aligned} \boldsymbol{F} &= \left(a_1 x_1^3 + a_2 x_1^2 x_2 + a_3 x_1 x_2^2 + a_4 x_2^3\right) \boldsymbol{e}_1 \\ &\quad + \left(a_5 x_1^3 + a_6 x_1^2 x_2 + a_7 x_1 x_2^2 + a_8 x_2^3\right) \boldsymbol{e}_2 \\ \boldsymbol{h} &= \left(b_1 x_1^3 + b_2 x_1^2 x_2 + b_3 x_1 x_2^2 + b_4 x_2^3\right) \boldsymbol{e}_1 \\ &\quad + \left(b_5 x_1^3 + b_6 x_1^2 x_2 + b_7 x_1 x_2^2 + b_8 x_2^3\right) \boldsymbol{e}_2 \\ \boldsymbol{g} &= \left(c_1 x_1^3 + c_2 x_1^2 x_2 + c_3 x_1 x_2^2 + c_4 x_2^3\right) \boldsymbol{e}_1 \\ &\quad + \left(c_5 x_1^3 + c_6 x_1^2 x_2 + c_7 x_1 x_2^2 + c_8 x_2^3\right) \boldsymbol{e}_2 \end{aligned}$$

Therefore,

$$\begin{aligned} b_1 &= \frac{a_1 - c_1}{\rho_1^3 - \rho_1}, \quad b_2 = \frac{a_2 - c_2}{\rho_1^2 \rho_2 - \rho_1}, \quad b_3 = \frac{a_3 - c_3}{\rho_1 \rho_2^2 - \rho_1}, \quad b_4 = \frac{a_4 - c_4}{\rho_2^3 - \rho_1}, \\ b_5 &= \frac{a_5 - c_5}{\rho_1^3 - \rho_2}, \quad b_6 = \frac{a_6 - c_6}{\rho_1^2 \rho_2 - \rho_2}, \quad b_7 = \frac{a_7 - c_7}{\rho_1 \rho_2^2 - \rho_2}, \quad b_8 = \frac{a_8 - c_8}{\rho_2^3 - \rho_2} \end{aligned}$$

Some of these coefficients are singular or near-singular if $\rho_i \approx 0$, $\rho_k \approx 1$, $\rho_i^2 \approx 1$, $\rho_1 \rho_2 \approx 1$, $\rho_2^3 \approx \rho_1$, and $\rho_1^3 \approx \rho_2$. These conditions are *resonance or near-resonance conditions of order three*. They can be written in the compact form

$$\rho_1^{m_1} \rho_2^{m_2} \approx \rho_i \quad \text{for} \quad i = 1 \text{ and } 2 \tag{3.45}$$

where $m_1 + m_2 = 3$.

Again, in the absence of resonances or near-resonances of order three, the b_i can be chosen to eliminate all of the nonlinear terms in (3.44) and the normal form of the map would be

$$\boldsymbol{y}_{k+1} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} \boldsymbol{y}_k \tag{3.46}$$

When $\rho_1^2 \approx 1$, b_1 and b_6 are singular or near-singular. Hence, $c_1 = a_1$ and $c_6 = a_6$ and the normal form of the nonlinear map would be

$$\boldsymbol{y}_{k+1} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} \boldsymbol{y}_k + \begin{bmatrix} a_1 y_{1k}^3 \\ a_6 y_{1k}^2 y_{2k} \end{bmatrix} \tag{3.47}$$

When $\rho_2^2 \approx 1$, b_3 and b_8 are singular or near-singular. Hence, $c_3 = a_3$ and $c_8 = a_8$ and the normal form of the nonlinear map would be

$$\boldsymbol{y}_{k+1} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} \boldsymbol{y}_k + \begin{bmatrix} a_3 y_{1k} y_{2k}^2 \\ a_8 y_{2k}^3 \end{bmatrix} \tag{3.48}$$

When $\rho_1 \rho_2 \approx 1$, b_2 and b_7 are singular or near-singular. Hence, $c_2 = a_2$ and $c_7 = a_7$ and the normal form of the nonlinear map would be

$$\boldsymbol{y}_{k+1} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} \boldsymbol{y}_k + \begin{bmatrix} a_2 y_{1k}^2 y_{2k} \\ a_7 y_{1k} y_{2k}^2 \end{bmatrix} \tag{3.49}$$

When $\rho_2^3 \approx \rho_1$, b_4 is singular or near-singular. Hence, $c_4 = a_4$ and the normal form of the nonlinear map would be

$$\mathbf{y}_{k+1} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} \mathbf{y}_k + \begin{bmatrix} a_4 y_{2k}^3 \\ 0 \end{bmatrix} \tag{3.50}$$

When $\rho_1^3 \approx \rho_2$, b_5 is singular or near-singular. Hence, $c_5 = a_5$ and the normal form of the nonlinear map would be

$$\mathbf{y}_{k+1} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} \mathbf{y}_k + \begin{bmatrix} 0 \\ a_5 y_{1k}^3 \end{bmatrix} \tag{3.51}$$

When $\rho_1 = e^{i\alpha}$, where $\alpha \neq 0$, $\rho_2 = e^{-i\alpha}$ and $y^{(2)}$ must be the complex conjugate of $y^{(1)}$. In this case, the resonance $\rho_1\rho_2 = 1$ occurs irrespective of the value of α; it is an *inevitable resonance*. The resonance conditions (3.45) yield the conditions

$$\rho_1^2 \approx 1 \quad \text{and} \quad \rho_1^4 \approx 1 \tag{3.52}$$

These two resonances are called *strong resonances*. In the absence of strong resonances, the normal form of the nonlinear map would be

$$\xi_{k+1} = e^{i\alpha}\xi_k + a_2\xi_k^2\bar{\xi}_k \tag{3.53}$$

In the presence of the strong resonance $\rho_1^4 = 1$ or $\rho_1 = e^{i\pi/2} = i$, the term $a_4\bar{\xi}^3$ is a resonance term, and the normal form would be

$$\xi_{k+1} = i\xi_k + a_2\xi_k^2\bar{\xi}_k + a_4\bar{\xi}_k^3 \tag{3.54}$$

where $\xi = y^{(1)}$. When $\rho = -i$, the normal form would be the complex conjugate of (3.54).

Example 3.6

We consider the map

$$\mathbf{x}_{k+1} = \begin{bmatrix} \rho_1 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} a_1 x_{1k}^3 + a_2 x_{2k}^2 x_{3k} \\ a_3 x_{1k}^2 x_{2k} + a_4 x_{3k}^3 \\ a_5 x_{1k} x_{2k}^2 + a_6 x_{2k}^3 \end{bmatrix}$$

In this case, $r = 3$ and

$$\begin{aligned} \mathbf{F} &= a_1 x_1^3 \mathbf{e}_1 + a_2 x_2^2 x_3 \mathbf{e}_1 + a_3 x_1^2 x_2 \mathbf{e}_2 + a_4 x_3^3 \mathbf{e}_2 + a_5 x_1 x_2^2 \mathbf{e}_3 + a_6 x_2^3 \mathbf{e}_3 \\ \mathbf{h} &= b_1 x_1^3 \mathbf{e}_1 + b_2 x_2^2 x_3 \mathbf{e}_1 + b_3 x_1^2 x_2 \mathbf{e}_2 + b_4 x_3^3 \mathbf{e}_2 + b_5 x_1 x_2^2 \mathbf{e}_3 + b_6 x_2^3 \mathbf{e}_3 \\ \mathbf{g} &= c_1 x_1^3 \mathbf{e}_1 + c_2 x_2^2 x_3 \mathbf{e}_1 + c_3 x_1^2 x_2 \mathbf{e}_2 + c_4 x_3^3 \mathbf{e}_2 + c_5 x_1 x_2^2 \mathbf{e}_3 + c_6 x_2^3 \mathbf{e}_3 \end{aligned}$$

Therefore,

$$b_1 = \frac{a_1 - c_1}{\rho_1^3 - \rho_1}, \quad b_2 = \frac{a_2 - c_2}{\rho_2^2\rho_3 - \rho_1}, \quad b_3 = \frac{a_3 - c_3}{\rho_1^2\rho_2 - \rho_2},$$
$$b_4 = \frac{a_4 - c_4}{\rho_3^3 - \rho_2}, \quad b_5 = \frac{a_5 - c_5}{\rho_1\rho_2^2 - \rho_3}, \quad b_6 = \frac{a_6 - c_6}{\rho_2^3 - \rho_3}$$

When the b_i are not singular or near-singular, the $c_i = 0$ and hence $\boldsymbol{g} = 0$. Consequently, all of the nonlinear terms can be eliminated and the normal form would be

$$\boldsymbol{y}_{k+1} = \begin{bmatrix} \rho_1 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{bmatrix} \boldsymbol{y}_k$$

When the ρ_i are such that one or more of the b_m are singular or near-singular, then b_m is arbitrary and $c_m = a_m$. For example, in the presence of only the resonance condition $\rho_2^2\rho_3 \approx \rho_1$, b_2 is arbitrary, $c_2 = a_2$, and the normal form would be

$$\boldsymbol{y}_{k+1} = \begin{bmatrix} \rho_1 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{bmatrix} \boldsymbol{y}_k + \begin{bmatrix} a_2 y_{2k}^2 y_{3k} \\ 0 \\ 0 \end{bmatrix}$$

3.3
Center-Manifold Reduction

We consider the local dynamics near a nonhyperbolic fixed point $\boldsymbol{x}^*$ of the nonlinear map (3.21), where $\boldsymbol{F}$ is an analytic vector function of $\boldsymbol{x}$. A fixed point of a map is called *hyperbolic* if none of the eigenvalues (multipliers) of its linearization at the fixed point is on the unit circle in the complex plane; otherwise, it is called *nonhyperbolic*. Whereas linearization is sufficient for ascertaining the stability of a hyperbolic fixed point according to the *Hartman–Grobman theorem*, linearization may not be sufficient for determining the stability of a nonhyperbolic fixed point. Moreover, according to the center-manifold theorem (Carr, 1981), there exists a C^r local center manifold for the nonlinear map (3.21) near $\boldsymbol{x}^*$. Furthermore, the long-time dynamics of (3.21) can be reduced to determining the dynamics on the center manifold. Next, we describe how to construct the center manifold in the neighborhood of a nonhyperbolic fixed point with one multiplier being equal to ± 1 with all of the other multipliers being inside or outside the unit circle. We assume that the fixed point has been shifted to the origin and that the linear part has been transformed into a Jordan canonical form; that is, we consider the map

$$\boldsymbol{x}_{k+1} = J\boldsymbol{x}_k + \boldsymbol{F}(\boldsymbol{x}) \tag{3.55}$$

We arrange the map (3.55) and rewrite it as

$$x_{k+1} = \rho x_k + f(x_k, \boldsymbol{y}_k) \tag{3.56}$$

$$\boldsymbol{y}_{k+1} = B\boldsymbol{y}_k + \boldsymbol{G}(x_k, \boldsymbol{y}_k) \tag{3.57}$$

where $\rho = \pm 1$, B is a constant matrix with none of its eigenvalues being on the unit circle, and f and $\boldsymbol{G}$ are scalar and vector-valued nonlinear functions of x_k and $\boldsymbol{y}_k$. According to the center-manifold theorem, there exists a center manifold

$$\boldsymbol{y} = \boldsymbol{h}(x) \tag{3.58}$$

Moreover, the dynamics of the map (3.56) and (3.57) is qualitatively similar to the dynamics on this manifold; that is,

$$x_{k+1} = \rho x_k + f\left[x_k, \boldsymbol{h}(x_k)\right] \tag{3.59}$$

Substituting (3.58) into (3.57) yields

$$\boldsymbol{h}(x_{k+1}) = B\boldsymbol{h}(x_k) + \boldsymbol{G}\left[x_k, \boldsymbol{h}(x_k)\right] \tag{3.60}$$

which upon using (3.59) becomes

$$\boldsymbol{h}\left\{\rho x + f\left[x, \boldsymbol{h}(x)\right]\right\} = B\boldsymbol{h}(x) + \boldsymbol{G}\left[x, \boldsymbol{h}(x)\right] \tag{3.61}$$

Using two examples, we describe how to construct approximate solutions of (3.61).

Example 3.7

We consider the map

$$x_{k+1} = x_k + \alpha_1 x_k y_k \tag{3.62}$$

$$y_{k+1} = \rho y_k + \alpha_2 x_k^2 \tag{3.63}$$

where $|\rho| < 1$ and α_1 and α_2 are constants. In this case, $B = \rho$, $f(x, y) = \alpha_1 x y$ and $G(x, y) = \alpha_2 x^2$ and (3.61) becomes

$$h\left[x + \alpha_1 x h(x)\right] = \rho h(x) + \alpha_2 x^2 \tag{3.64}$$

We seek an approximate solution of (3.64) in the form

$$h(x) = \Gamma_2 x^2 + \Gamma_3 x^3$$

and obtain

$$\Gamma_2\left(x + \alpha_1\Gamma_2 x^3 + \alpha_1\Gamma_3 x^4\right)^2 + \Gamma_3\left(x + \alpha_1\Gamma_2 x^3 + \alpha_1\Gamma_3 x^4\right)^3$$
$$= \rho\Gamma_2 x^2 + \rho\Gamma_3 x^3 + \alpha_2 x^2 + \cdots$$

or

$$\Gamma_2 x^2 + \Gamma_3 x^3 - \rho\Gamma_2 x^2 - \rho\Gamma_3 x^3 - \alpha_2 x^2 + \cdots = 0 \tag{3.65}$$

Equating to zero each of the coefficients of x^2 and x^3 in (3.65) and solving the resulting algebraic equations, we obtain

$$\Gamma_2 = \frac{\alpha_2}{1-\rho} \quad \text{and} \quad \Gamma_3 = 0$$

Hence, the center manifold is given by

$$h(x) = \frac{\alpha_2}{1-\rho} x^2 + \cdots \tag{3.66}$$

and it follows from (3.62) that the long-time dynamics on this center manifold is given by

$$x_{k+1} = x_k + \frac{\alpha_1\alpha_2}{1-\rho} x_k^3 + \cdots \tag{3.67}$$

We note that linearization is not sufficient for determining the stability of the origin because the multipliers are $\rho_1 = 1$ and $\rho_2 < 1$. However, including the nonlinear terms, we find from (3.67) that the origin is unstable when $\alpha_1\alpha_2 > 0$ and stable when $\alpha_1\alpha_2 < 0$.

Example 3.8

We consider the map

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} + \begin{bmatrix} a_1 x_k^2 + a_2 x_k y_k + a_3 x_k z_k \\ a_4 x_k^2 + a_5 x_k y_k + a_6 x_k z_k \\ a_7 x_k^2 + a_8 x_k y_k + a_9 x_k z_k \end{bmatrix} \tag{3.68}$$

Clearly, the origin is a nonhyperbolic fixed point and the multipliers associated with this point are $-1, 1/2$, and $1/4$. Because linearization is not sufficient to ascertain the stability of this fixed point, we investigate the dynamics on the center manifold.

In this case,

$$\rho = -1 \quad \text{and} \quad B = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$$

and (3.59) and (3.61) become

$$x_{k+1} = -x_k + a_1 x_k^2 + a_2 x_k h_1(x_k) + a_3 x_k h_2(x_k) \tag{3.69}$$

$$\begin{aligned} &\boldsymbol{h}\left[-x + a_1 x^2 + a_2 x h_1(x) + a_3 x h_2(x)\right] \\ &= \begin{bmatrix} \frac{1}{2} h_1(x) \\ \\ \frac{1}{4} h_2(x) \end{bmatrix} + \begin{bmatrix} a_4 x^2 + a_5 x h_1(x) + a_6 x h_2(x) \\ a_7 x^2 + a_8 x h_1(x) + a_9 x h_2(x) \end{bmatrix} \end{aligned} \tag{3.70}$$

where h_1 and h_2 are the components of $\boldsymbol{h}$.

Next, we seek an approximate solution for $\boldsymbol{h}(x)$ in power series of x; that is,

$$h_1(x) = b_1 x^2 + \cdots \quad \text{and} \quad h_2(x) = b_2 x^2 + \cdots \tag{3.71}$$

Substituting (3.71) into (3.70), expanding the result in Taylor series for small x, and keeping up to quadratic terms, we obtain

$$b_1 x^2 = \tfrac{1}{2} b_1 x^2 + a_4 x^2 + \cdots$$

$$b_2 x^2 = \tfrac{1}{4} b_2 x^2 + a_7 x^2 + \cdots$$

Therefore, $b_1 = 2a_4$, $b_2 = \frac{4}{3} a_7$, and hence

$$h_1(x) = 2a_4 x^2 + \cdots \quad \text{and} \quad h_2(x) = \tfrac{4}{3} a_7 x^2 + \cdots \tag{3.72}$$

Substituting (3.72) into (3.69), we obtain

$$x_{k+1} = -x_k + a_1 x_k^2 + \left(2a_2 a_4 + \tfrac{4}{3} a_3 a_7\right) x_k^3 \tag{3.73}$$

for the dynamics on the center manifold.

We note that the problem of determining the stability of the origin of (3.68) has been reduced to determining the stability of the origin of the map (3.73). To ascertain this stability, we introduce the near-identity transformation

$$x = \xi + b_3 \xi^2 \tag{3.74}$$

into (3.73) and choose b_3 to eliminate the quadratic term. The result is $b_3 = 1/2a_1$ and

$$\xi_{k+1} = -\xi_k + \left(a_1^2 + 2a_2 a_4 + \tfrac{4}{3} a_3 a_7\right) \xi_k^3 + \cdots \tag{3.75}$$

Therefore, the origin is stable or unstable depending on whether

$$a_1^2 + 2a_2 a_4 + \tfrac{4}{3} a_3 a_7$$

is positive or negative.

3.4
Local Bifurcations

As a single control parameter is varied, an asymptotically stable fixed point of a map can experience a bifurcation if it becomes nonhyperbolic. There are three cases in which a fixed point $\boldsymbol{x} = \boldsymbol{x}^*$ of the map (3.21) ceases to be hyperbolic at a certain critical value $\mu = \mu_c$ of the control parameter. These cases are:

1. $D_{\boldsymbol{x}} \boldsymbol{F}(\boldsymbol{x}^*; \mu_c)$ has one eigenvalue equal to 1, with the remaining $(n-1)$ eigenvalues being within the unit circle.
2. $D_{\boldsymbol{x}} \boldsymbol{F}(\boldsymbol{x}^*; \mu_c)$ has one eigenvalue equal to -1, with the remaining $(n-1)$ eigenvalues being within the unit circle.
3. $D_{\boldsymbol{x}} \boldsymbol{F}(\boldsymbol{x}^*; \mu_c)$ has a pair of complex conjugate eigenvalues on the unit circle, with the remaining $(n-2)$ eigenvalues being within the unit circle.

According to the center-manifold theorem, analysis of the dynamics of an n-dimensional map near a nonhyperbolic fixed point can be reduced to the analysis of the dynamics on the center manifold. The analysis can be reduced to a one-dimensional map in cases 1 and 2 and to a two-dimensional map in case 3. In case 1, three types of bifurcation can occur: fold (saddle-node, tangent), transcritical, and pitchfork bifurcations; in case 2, flip or period-doubling bifurcation can occur; and in case 3, Hopf (Neimark–Sacker) bifurcation can occur. For more information on bifurcation analyses for maps, the reader is referred to the books of Arnold (1988), Guckenheimer and Holmes (1983), Iooss (1979), Wiggins (1990), and Nayfeh and Balachandran (1995). Next, we discuss these five types of bifurcation.

3.4.1
Fold or Tangent or Saddle-Node Bifurcation

This type of bifurcation can occur when an asymptotically stable fixed point looses stability due to an eigenvalue exiting the unit circle through $+1$ as a control parameter exceeds a critical value. In this case, analysis of the dynamics of the n-dimensional map can be reduced to the analysis of a one-dimensional map; that is,

$$x_{k+1} = f(x_k; \mu) \tag{3.76}$$

We assume that the fixed point is at $x = 0$ when $\mu = 0$ and that $f(0,0) = 0$ and $f_x(0,0) = 1$.

To investigate the dynamics of (3.76), we expand $f(x;\mu)$ in a Taylor series for small x and μ and obtain

$$\begin{aligned} x_{k+1} = x_k &+ f_\mu \mu + \tfrac{1}{2}\left(f_{\mu\mu}\mu^2 + 2 f_{x\mu} x_k \mu + f_{xx} x_k^2\right) \\ &+ \tfrac{1}{6}\left(f_{\mu\mu\mu}\mu^3 + 3 f_{x\mu\mu} x_k \mu^2 + 3 f_{xx\mu} x_k^2 \mu + f_{xxx} x_k^3\right) + \cdots \end{aligned} \tag{3.77}$$

As $x \to 0$ and $\mu \to 0$, the limit of (3.77) depends on whether f_μ and f_{xx} are equal to or different from zero. In the latter case, (3.77) tends to

$$x_{k+1} = x_k + f_\mu \mu + \tfrac{1}{2} f_{xx} x_k^2 \tag{3.78}$$

The fixed points of (3.78) are

$$x^* = \pm \sqrt{\frac{-2 f_\mu \mu}{f_{xx}}} \tag{3.79}$$

The multipliers associated with these two fixed points are

$$\rho = 1 \pm f_{xx} \sqrt{\frac{-2 f_\mu \mu}{f_{xx}}} \tag{3.80}$$

It follows from (3.79) that there are two branches of fixed points in the neighborhood of $(x, \mu) = (0, 0)$ for $\mu < 0$ if $f_\mu f_{xx} > 0$ and for $\mu > 0$ if $f_\mu f_{xx} < 0$. Then, it follows from (3.80) that the upper branch is stable and the lower branch is unstable if $f_{xx} < 0$ and that the upper branch is unstable and the lower branch is stable if $f_{xx} > 0$. This bifurcation of the nonhyperbolic fixed point at the origin as μ passes through zero is called *fold* or *tangent* or *saddle-node bifurcation*.

Example 3.9

We consider the one-dimensional map

$$x_{k+1} = x_k + \mu - x_k^2 \tag{3.81}$$

where μ is a scalar control parameter. When $\mu > 0$, it is clear from Figure 3.1a that this map intersects the identity map $x_{k+1} = x_k$ in two points. In other words, for $\mu > 0$, (3.81) has the nontrivial fixed points

$$x_1^* = \sqrt{\mu} \quad \text{and} \quad x_2^* = -\sqrt{\mu}$$

The Jacobian matrix associated with the fixed point x_{j0} has the single eigenvalue

$$\rho = 1 - 2x_j^*$$

The fixed point x_2^* is an unstable node for all $\mu > 0$ because $|\rho| > 1$. On the other hand, the fixed point x_1^* is a stable node for $0 < \mu < 1$ because $|\rho| < 1$. When μ is decreased to zero, the two fixed points approach each other and coalesce at the single fixed point $x = 0$, as shown in Figure 3.1b. The map (3.81) is tangent to the identity map and, hence, the associated bifurcation is called *tangent bifurcation*. When $\mu < 0$, the map (3.81) does not intersect the identity map, as shown in Figure 3.1c, and hence for $\mu < 0$, (3.81) does not have any fixed points. In Figure 3.2, we show the different fixed points of (3.81) and their stability in the vicinity of the origin of the $x - \mu$ space. Broken and solid lines are used to represent branches

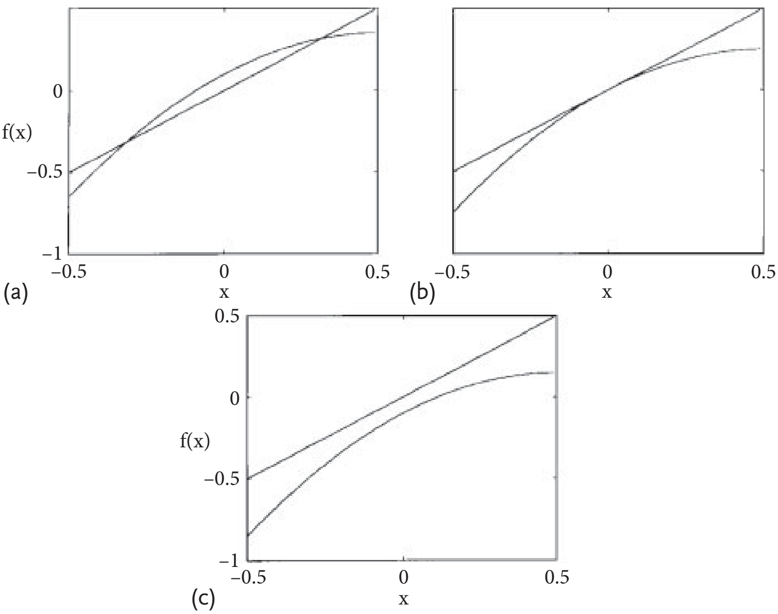

Figure 3.1 The functions $f(x) = \mu + x - x^2$ and x for (a) $\mu = 0.1$, (b) $\mu = 0.0$, and (c) $\mu = -0.1$.

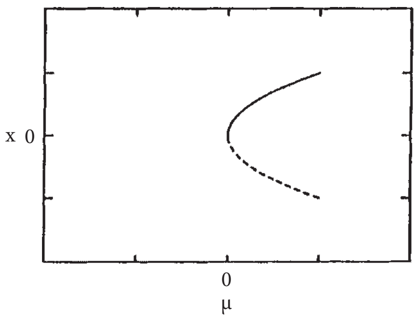

Figure 3.2 Scenario in the vicinity of a saddle-node bifurcation.

of unstable and stable fixed points, respectively. A *saddle-node* or *tangent bifurcation* occurs at $(x, \mu) = (0, 0)$.

Diagrams such as Figure 3.2 in which the variation of solutions and their stability are displayed in the state-control space are called *bifurcation diagrams*. In the bifurcation diagram, a branch of stable solutions is called a *stable branch* and a branch of unstable solutions is called an *unstable branch*. In most situations, a branch of solutions either ends or begins at a bifurcation point.

3.4.2 Transcritical Bifurcation

When $f_\mu = 0$ and $f_{xx} \neq 0$, the limit of (3.77) as $x \to 0$ and $\mu \to 0$ is

$$x_{k+1} = x_k + \tfrac{1}{2}\left(f_{\mu\mu}\mu^2 + 2f_{x\mu}x_k\mu + f_{xx}x_k^2\right) \tag{3.82}$$

whose fixed points are

$$x_1^* = \frac{-f_{x\mu} + \sqrt{f_{x\mu}^2 - f_{\mu\mu}f_{xx}}}{f_{xx}}\mu \quad \text{and} \quad x_2^* = \frac{-f_{x\mu} - \sqrt{f_{x\mu}^2 - f_{\mu\mu}f_{xx}}}{f_{xx}}\mu \tag{3.83}$$

provided that $f_{x\mu}^2 - f_{\mu\mu}f_{xx} > 0$. It follows from (3.83) that there are two curves of fixed points in the neighborhood of $(x, \mu) = (0, 0)$, which intersect transversely at $(0, 0)$. The multipliers associated with these fixed points are

$$\rho(x_1^*) = 1 + \sqrt{f_{x\mu}^2 - f_{\mu\mu}f_{xx}}\,\mu \quad \text{and} \quad \rho(x_2^*) = 1 - \sqrt{f_{x\mu}^2 - f_{\mu\mu}f_{xx}}\,\mu \tag{3.84}$$

Therefore, x_1^* is stable when $\mu < 0$ and unstable when $\mu > 0$. On the other hand, x_2^* is unstable when $\mu < 0$ and stable when $\mu > 0$. In this case, we have two branches of fixed points that interchange stability at $(0, 0)$. The bifurcation of the nonhyperbolic fixed point at the origin as μ passes through zero is called *transcritical bifurcation.*

Example 3.10

We consider the one-dimensional map

$$x_{k+1} = x_k + \mu x_k - x_k^2 \tag{3.85}$$

where μ is again a scalar control parameter. This map has the two fixed points

$$\begin{aligned} x_1^* &= 0: \quad \text{trivial fixed point} \\ x_2^* &= \mu: \quad \text{nontrivial fixed point}\,. \end{aligned}$$

For the fixed point x_j^*, the Jacobian matrix has the single eigenvalue

$$\rho = 1 + \mu - 2x_j^*$$

Hence, it follows that the trivial fixed point is stable for $-2 < \mu < 0$ and unstable for all $\mu > 0$. On the other hand, the nontrivial fixed point is unstable for all $\mu < 0$ and stable for $0 < \mu < 2$. The scenario in the vicinity of $(x, \mu) = (0, 0)$ is illustrated in Figure 3.3.

A *transcritical bifurcation* occurs at the origin.

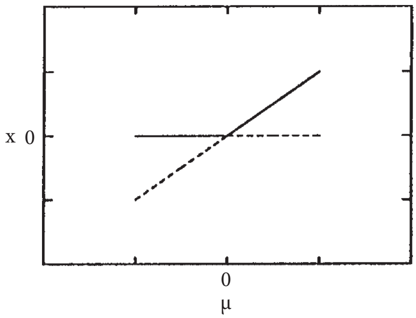

Figure 3.3 Scenario in the vicinity of a transcritical bifurcation.

3.4.3
Pitchfork Bifurcation

When $f_\mu = 0$, $f_{xx} = 0$, $f_{x\mu} \neq 0$, and $f_{xxx} \neq 0$, the limit of (3.77) as $x \to 0$ and $\mu \to 0$ is

$$x_{k+1} = x_k + f_{x\mu} x_k \mu + \tfrac{1}{6} f_{xxx} x_k^3 \tag{3.86}$$

whose fixed points are

$$x_1^* = 0\,, \quad x_2^* = \sqrt{\frac{-6 f_{x\mu}\mu}{f_{xxx}}}\,, \quad \text{and} \quad x_3^* = -\sqrt{\frac{-6 f_{x\mu}\mu}{f_{xxx}}} \tag{3.87}$$

It follows from (3.87) that there are three branches of solutions. The first branch exists for all values of μ and the other two branches exist for $\mu > 0$ if $f_{x\mu} f_{xxx} < 0$ and for $\mu < 0$ if $f_{x\mu} f_{xxx} > 0$. The multipliers associated with these fixed points are

$$\rho(x_1^*) = 1 + f_{x\mu}\mu\,, \quad \rho(x_2^*) = 1 - 2 f_{x\mu}\mu\,, \quad \text{and} \quad \rho(x_3^*) = 1 - 2 f_{x\mu}\mu \tag{3.88}$$

When $f_{x\mu} > 0$, the trivial fixed point is stable when $\mu < 0$ and unstable when $\mu > 0$. When $f_{xxx} < 0$, the two nontrivial fixed points exist and are stable when $\mu > 0$. This bifurcation of the nonhyperbolic fixed point at the origin as μ passes through zero is called *supercritical pitchfork bifurcation*. On the other hand, when $f_{xxx} > 0$, the two nontrivial fixed points exist and are unstable when $\mu < 0$. This bifurcation of the nonhyperbolic fixed point at the origin as μ passes through zero is called *subcritical or reverse pitchfork bifurcation*.

Example 3.11

We consider the one-dimensional map

$$x_{k+1} = x_k + \mu x_k + \alpha x_k^3 \tag{3.89}$$

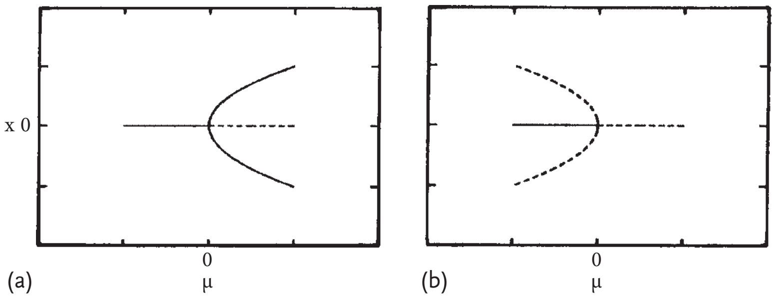

Figure 3.4 Local scenarios: (a) supercritical pitchfork bifurcation and (b) subcritical pitchfork bifurcation.

where μ is a scalar control parameter. This map has the fixed points

$$x_1^* = 0: \quad \text{trivial fixed point}$$
$$x_{2,3}^* = \pm\sqrt{-\mu/\alpha}: \quad \text{nontrivial fixed points.}$$

The Jacobian matrix associated with the fixed point x_j^* has the single eigenvalue

$$\rho = 1 + \mu + 3\alpha x_j^{*2}$$

Therefore, the trivial fixed point is stable for $-2 < \mu < 0$ and unstable for all $\mu > 0$. For $\alpha < 0$, nontrivial fixed points exist only for $\mu > 0$, and they are stable for $0 < \mu < 1$. For $\alpha > 0$, nontrivial fixed points exist only for $\mu < 0$, and they are unstable. The scenarios for $\alpha = -1$ and $\alpha = 1$ near the origin $(x, \mu) = (0, 0)$ are shown in Figure 3.4a and b, respectively. There is a *supercritical pitchfork bifurcation* at the origin in Figure 3.4a and a *subcritical pitchfork bifurcation* at the origin in Figure 3.4b.

3.4.4
Flip or Period-Doubling Bifurcation

As in the cases of fold, transcritical, and pitchfork bifurcations, analysis of the dynamics of an n-dimensional map in the neighborhood of a nonhyperbolic fixed point with one eigenvalue being equal to -1 and the remaining eigenvalues being inside the unit circle can be reduced, by using the center-manifold theorem, to the analysis of the dynamics of the following one-dimensional map on the center manifold:

$$x_{k+1} = f(x_k; \mu) \tag{3.90}$$

We assume that the fixed point is at $x = 0$ when $\mu = 0$ and that $f(0, 0) = 0$ and $f_x(0, 0) = -1$.

Again, we expand $f(x;\mu)$ in a Taylor series for small x and μ and obtain

$$x_{k+1} = -x_k + f_\mu \mu + \tfrac{1}{2}\left(f_{\mu\mu}\mu^2 + 2f_{x\mu}x_k\mu + f_{xx}x_k^2\right) + \tfrac{1}{6}\left(f_{\mu\mu\mu}\mu^3 + 3f_{x\mu\mu}x_k\mu^2 + 3f_{xx\mu}x_k^2\mu + f_{xxx}x_k^3\right) + \cdots \tag{3.91}$$

Therefore, the fixed points of the map (3.91) are solutions of

$$-2x + f_\mu \mu + \tfrac{1}{2}\left(f_{\mu\mu}\mu^2 + 2f_{x\mu}x\mu + f_{xx}x^2\right) + \tfrac{1}{6}\left(f_{\mu\mu\mu}\mu^3 + 3f_{x\mu\mu}x\mu^2 + 3f_{xx\mu}x^2\mu + f_{xxx}x^3\right) + \cdots = 0 \tag{3.92}$$

For (x,μ) near $(0,0)$, (3.92) has a single solution x^* given by

$$x^* = \tfrac{1}{2} f_\mu \mu + O(\mu^2) \tag{3.93}$$

To ascertain the stability of this fixed point, we differentiate (3.91) with respect to x and obtain

$$D_x f(x;\mu) = -1 + f_{x\mu}\mu + f_{xx}x + \tfrac{1}{2} f_{x\mu\mu}\mu^2 + f_{xx\mu}x\mu + \tfrac{1}{2} f_{xxx}x^2 + \cdots \tag{3.94}$$

Substituting (3.93) into (3.94) yields the multiplier

$$\rho(x^*) = -1 + \left(f_{x\mu} + \tfrac{1}{2} f_\mu f_{xx}\right)\mu + O(\mu^2) \tag{3.95}$$

Hence, if $f_{x\mu} + 1/2 f_\mu f_{xx} > 0$, the fixed point x^* is stable when $\mu > 0$ and unstable when $\mu < 0$. On the other hand, if $f_{x\mu} + 1/2 f_\mu f_{xx} < 0$, the fixed point x^* is stable when $\mu < 0$ and unstable when $\mu > 0$.

For nondegenerate bifurcation,

$$\frac{d\rho}{d\mu}(x^*) = f_{x\mu} + \frac{1}{2} f_\mu f_{xx} \neq 0 \tag{3.96}$$

In other words, the multiplier of the map at the fixed point should exit the unit circle through -1 with nonzero speed.

To investigate the bifurcating solutions, we first introduce a transformation to reduce (3.91) to its normal form by shifting the fixed point to $x = 0$ and eliminating the quadratic term. To this end, we let

$$x = \tfrac{1}{2} f_\mu \mu + y + b y^2 \tag{3.97}$$

Substituting (3.97) into (3.91) and choosing b to eliminate the quadratic term in y, we obtain the normal form

$$y_{k+1} = -(1+\nu)y_k + \alpha y_k^3 \tag{3.98}$$

where

$$\nu = -\left(f_{x\mu} + \tfrac{1}{2} f_\mu f_{xx}\right)\mu + O(\mu^2)\,, \quad b = \tfrac{1}{4} f_{xx} + O(\mu)\,,$$
$$\alpha = \tfrac{1}{6} f_{xxx} + \tfrac{3}{8} f_{xx}^2 + O(\mu)$$

The fixed point $y = 0$ of (3.98) ceases to be hyperbolic when $\nu = 0$ and the map undergoes a flip bifurcation there. To determine the bifurcating period-two orbit, we investigate the fixed points of the second iterate of the map (3.98); that is,

$$y_{k+2} = (1 + 2\nu)y_k - 2\alpha y_k^3 + \cdots \tag{3.99}$$

It follows from (3.99) that the fixed points of the second iterate map are

$$y^* = 0\,, \sqrt{\frac{\nu}{\alpha}}\,, -\sqrt{\frac{\nu}{\alpha}}$$

The first fixed point is also a fixed point of the map. The associated multipliers are

$$\rho = 1 + 2\nu\ , 1 - 4\nu\ , 1 - 4\nu$$

Therefore, the fixed point $y^* = 0$ is stable when $\nu < 0$ and unstable when $\nu > 0$. If $\alpha > 0$, the nontrivial fixed points exist and are stable for $\nu > 0$. Consequently, the period-two bifurcating orbit is stable, and the bifurcation of the nonhyperbolic fixed point at the origin as μ passes through zero is called *supercritical period-doubling or flip bifurcation*. If $\alpha < 0$, the nontrivial fixed points exist and are unstable for $\nu < 0$. Consequently, the period-two bifurcating orbit is unstable, and the bifurcation of the nonhyperbolic fixed point at the origin as μ passes through zero is called *subcritical period-doubling or flip bifurcation*.

Example 3.12

We consider the logistic map

$$x_{k+1} = F(x_k) = 4\alpha x_k(1 - x_k) \tag{3.100}$$

for $0 \leq x \leq 1$ and positive α. The fixed points of this map are

$$x_1^* = 0 \quad \text{and} \quad x_2^* = 1 - \frac{1}{4\alpha}$$

The multiplier associated with the fixed point x_j^* is given by

$$\rho_j = 4\alpha(1 - 2x_j^*)$$

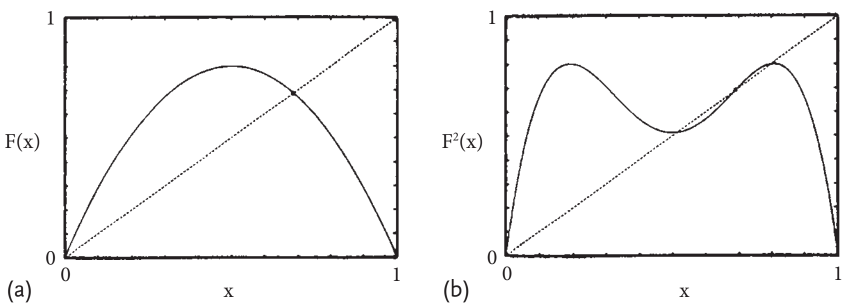

Figure 3.5 Graphs to determine solutions of the logistic equation at $\alpha = 0.8$: (a) fixed points and (b) period-two points.

The trivial fixed point x_1^* exists for all α. Because $\rho_1 = 4\alpha$, it is stable when $\alpha < 1/4$. The nontrivial fixed point x_2^* exists for $\alpha > 1/4$. Because $\rho_2 = 2 - 4\alpha$, this nontrivial fixed point is stable for $1/4 < \alpha < 3/4$. At $\alpha = 3/4$, x_2^* is nonhyperbolic because $\rho_2 = -1$ and it is unstable for $\alpha > 3/4$. In Figure 3.5a, we plot $F(x)$ versus x when $\alpha = 0.8$. The intersections of the line $y = x$ with the curve $F(x)$ give the fixed points of $F(x)$. They occur at $x_1^* = 0$ and $x_2^* = 0.6875$. The fixed point $x_2^* = 0.6875$ is unstable because $|F'(x_2^* = 0.6875)| = 1.2 > 1$.

Next, we investigate the fixed points of $F^{(2)}(x)$, which are given by

$$F^{(2)}(x) = F[F(x)] = F[4\alpha x(1-x)]$$

or

$$F^{(2)}(x) = 16\alpha^2 x(1-x)\left[1 - 4\alpha x(1-x)\right]$$

Hence, the fixed points of $F^{(2)}(x)$ are given by the solutions of

$$16\alpha^2 x(1-x)\left[1 - 4\alpha x(1-x)\right] = x$$

which are

$$x^* = 0\,, 1 - \frac{1}{4\alpha}\,, \quad \text{and} \quad \frac{1}{2} + \frac{1}{4\alpha}\left[\frac{1}{2} \pm \sqrt{\left(2\alpha - \frac{1}{2}\right)^2 - 1}\right]$$

The first two fixed points are also fixed points of $F(x)$. In Figure 3.5b, we plot $F^{(2)}(x)$ versus x when $\alpha = 0.8$. The intersections of the curve $F^{(2)}(x)$ with the curve $y = x$ give the fixed points of $F^{(2)}(x)$. We note that there are four intersections. The dot corresponds to the fixed point $x^* = 1 - 1/(4\alpha) = 0.6875$ of $F^{(2)}(x)$, which is also a fixed point of $F(x)$. The other three fixed points are $x^* = 0$, $x^* = 0.5130$, and $x^* = 0.7995$. We note that

$$F(0.5130) = 0.7995 \quad \text{and} \quad F^{(2)}(0.5130) = 0.5130$$

and that

$$F(0.7995) = 0.5130 \quad \text{and} \quad F^{(2)}(0.7995) = 0.7995$$

Hence, $x^* = 0.5130$ and $x^* = 0.7995$ are period-two points of $F(x)$. To determine the stability of the fixed points x_j^* of $F^{(2)}(x)$, we calculate its Jacobian; that is,

$$\rho_j = \frac{d}{dx}\left[F^{(2)}(x_j^*)\right] = 16\alpha^2(1 - 2x_j^*)\left[1 - 8\alpha x_j^*(1 - x_j^*)\right]$$

For the fixed points $x^* = 0$ and $x^* = 0.6875$, $\rho = 10.24$ and $\rho = 1.44$, respectively. Hence, these fixed points are unstable. This is expected because these fixed points are unstable fixed points of $F(x)$; $F'(x = 0) = 3.2$ and $F'(x = 0.6875) = -1.2$. For the fixed points $x^* = 0.5130$ and $x^* = 0.7995$ of $F^{(2)}(x)$, $\rho = 0.16$, and hence both of the period-two points of $F(x)$ are stable.

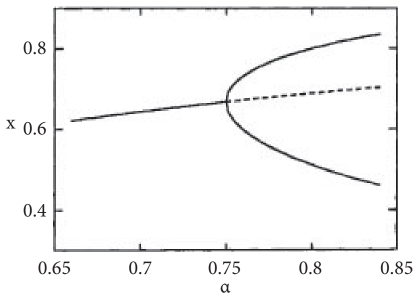

Figure 3.6 Scenario in the vicinity of a period-doubling bifurcation of the fixed point of the logistic map.

We numerically determined that the fixed points x_3^* and x_4^* of $F^{(2)}(x)$ or equivalently the period-two points of $F(x)$ are stable for $\alpha < 0.85$. In Figure 3.6, we show the different solutions of $F(x)$ and their stability for $0.65 < \alpha < 0.85$. The branches of stable and unstable solutions are depicted by solid and broken lines, respectively. The fixed point x_2^* of $F(x)$ experiences a *supercritical period-doubling bifurcation* at $\alpha = 0.75$. As a consequence, two branches of period-two points emerge from this bifurcation point. We note that an iterate of $F(x)$ initiated at either of the two period-two points flips back and forth between them because

$$F(x_3^*) = x_4^* \quad \text{and} \quad F(x_4^*) = x_3^*$$

For this reason, the period-doubling bifurcation of a fixed point of a map is also called a *flip bifurcation*.

The fixed point x_2^* and the period-two points x_3^* and x_4^* of the map $F(x)$ are all fixed points of the map $F^{(2)}(x)$. At $\alpha = 3/4$, the fixed point $x_2^* = 2/3$ of $F^{(2)}(x)$ is nonhyperbolic because $\rho_2 = 1$. Hence, from Figure 3.6, we infer that this fixed point of $F^{(2)}(x)$ experiences a pitchfork bifurcation at $\alpha = 3/4$. In this case, the pitchfork bifurcation is supercritical. In other cases, it is possible that a period-doubling bifurcation of $F(x)$ can correspond to a subcritical pitchfork bifurcation of the map $F^{(2)}(x)$. The associated period-two points that arise due to this bifurcation will be unstable.

3.4.5
Hopf or Neimark–Sacker Bifurcation

When a fixed point of (3.55) is nonhyperbolic with a pair of complex conjugate eigenvalues on the unit circle, the fixed point of the map can experience what is called the *Neimark–Sacker bifurcation* or *Hopf bifurcation* (Iooss, 1979; Wiggins, 1990). This bifurcation can occur in two- and higher-dimensional maps. Again, center-manifold reduction analysis can be used to reduce the analysis of the dynamics of the n-dimensional map in the vicinity of the nonhyperbolic fixed point into the analysis of the dynamics of a two-dimensional map on the center manifold (Carr, 1981); that is,

$$x_{k+1} = f(x_k, y_k; \mu) \tag{3.101}$$

$$y_{k+1} = g(x_k, y_k; \mu) \tag{3.102}$$

We assume that coordinate and parameter transformations have been used so that the fixed point is at $(x, y) = (0, 0)$ when $\mu = 0$. Hence,

$$f(0, 0; 0) = 0\,, \quad g(0, 0; 0) = 0$$

and the Jacobian matrix

$$A = \begin{bmatrix} \frac{\partial f}{\partial x}(0,0;0) & \frac{\partial f}{\partial y}(0,0;0) \\ \frac{\partial g}{\partial x}(0,0;0) & \frac{\partial g}{\partial y}(0,0;0) \end{bmatrix} \tag{3.103}$$

has a pair of complex-conjugate eigenvalues $\rho = e^{i\beta}$ and $\bar{\rho} = e^{-i\beta}$ lying on the unit circle.

In the absence of strong resonances (i.e., $\rho^n \neq 1$ for $n = 1, 2, 3, 4$), one can follow steps similar to those in Examples 3.4 and 3.5 and the next example to obtain the following normal form of the map (3.101) and (3.102):

$$z_{k+1} = \left(e^{i\beta} + \nu\mu e^{i\tau}\right) z_k + \alpha z^2 \bar{z} \tag{3.104}$$

where ν and τ are real-valued parameters and z is a complex-valued quantity. We assume that $\nu \cos(\beta - \tau) \neq 0$ so that the pair of complex-conjugate multipliers transversely exit the unit circle as μ varies past zero. Substituting the polar form

$$z = re^{i\theta} \tag{3.105}$$

into (3.104) and separating real and imaginary parts, we obtain

$$r_{k+1} = r_k + \nu\mu \cos(\beta - \tau) r_k + \Lambda_r r_k^3 + \cdots \tag{3.106}$$

$$\theta_{k+1} = \theta_k + \beta - \nu\mu \sin(\beta - \tau) + \Lambda_i r_k^2 + \cdots \tag{3.107}$$

for small r and μ, where $\Lambda_r + i\Lambda_i = \alpha e^{-i\beta}$.

We note that, because the r component is independent of θ, the problem is reduced to studying the stability of the fixed points of the one-dimensional map (3.106). We assume that $\Lambda_r \neq 0$; otherwise higher-order terms need to be included in (3.104). There are three fixed points:

$$r = 0\,, \quad r = \sqrt{\frac{-\nu\mu\cos(\beta - \tau)}{\Lambda_r}}\,, \quad \text{and} \quad -\sqrt{\frac{-\nu\mu\cos(\beta - \tau)}{\Lambda_r}} \tag{3.108}$$

The origin is asymptotically stable for $\nu\mu\cos(\beta - \tau) < 0$, unstable for $\nu\mu\cos(\beta - \tau) > 0$, unstable for $\nu\mu\cos(\beta - \tau) = 0$ and $\Lambda_r > 0$, and asymptotically stable for $\nu\mu\cos(\beta-\tau)=0$ and $\Lambda_r<0$. On the other hand, the nontrivial fixed points exist when $-\nu\mu\Lambda_r\cos(\beta-\tau)>0$ and correspond to *invariant circles*. They are stable for $\nu\mu\cos(\beta-\tau)>0$ and $\Lambda_r<0$ and unstable for $\nu\mu\cos(\beta-\tau)<0$ and $\Lambda_r>0$.

Substituting for the nontrivial fixed points from (3.108) into (3.107) leads to the circle map

$$\theta_{k+1}=\theta_k+\beta-\nu\mu\sin(\beta-\tau)-\frac{\nu\mu\Lambda_i\cos(\beta-\tau)}{\Lambda_r} \tag{3.109}$$

which describes the dynamics on the invariant circle. The dynamics on the invariant circle is periodic or aperiodic (densely fills the invariant circle) depending on whether

$$\chi=\beta-\nu\mu\sin(\beta-\tau)-\frac{\nu\mu\Lambda_i\cos(\beta-\tau)}{\Lambda_r}$$

is rational or irrational. A *rational number* is a number that can be written as the ratio of two integers; that is, $\chi=p/q$ where p and q are integers. An *irrational number* cannot be written as the ratio of two integer. For example, 0.2 is a rational number because it can be written as 1/5, whereas $\sqrt{5}$ is an irrational number.

When $\chi=p/q$ where p and q are integers, the orbit of (3.109) starting at θ_0 consists of q points

$$\{\theta_0,\theta_0+2\pi\chi,\theta_0+4\pi\chi,\cdots,\theta_0+2(q-2)\pi\chi,\theta_0+2(q-1)\pi\chi\}$$

We note that the next point in the orbit is $\theta_0+2q\pi\chi=\theta_0+2p\pi=\theta_0$. Consequently, the orbit would repeat itself. This orbit can be represented by q points on a circle. For example, when $\chi=3/5$, the orbit consists of the five points

$$\left\{\theta_0,\theta_0+\tfrac{6}{5}\pi,\theta_0+\tfrac{12}{5}\pi,\theta_0+\tfrac{18}{5}\pi,\theta_0+\tfrac{24}{5}\pi\right\}$$

which can be represented as five points on a circle, as shown in Figure 3.7a.

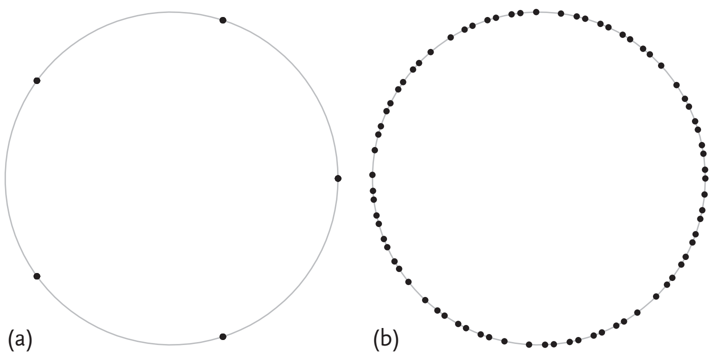

Figure 3.7 Dynamics of a linear circle map: (a) periodic motion when $\chi=3/5$ and (b) quasi-periodic motion when $\chi=1/\sqrt{7}$.

When χ is irrational, starting from any point θ_0 on a circle, we find that none of the iterates $F^{(m)}$ returns to the initial point θ_0 for any finite value of m. Thus, the orbit wanders on the circle, filling it up without becoming periodic. In other words, the orbit for irrational χ is aperiodic and is an example of a *quasiperiodic orbit*. An example is shown in Figure 3.7b for $\chi = 1/\sqrt{7}$. We note that two quasiperiodic orbits starting from two points a small distance apart remain close for all subsequent iterations; that is, there is no sensitivity to initial conditions.

Example 3.13

We consider the map

$$x_{k+1} = (1+\mu)y_k \tag{3.110}$$

$$y_{k+1} = y_k - x_k - 2x_k y_k \tag{3.111}$$

which has the fixed points $(x, y) = (0, 0)$ and $(x, y) = (-1/2 - 1/2\mu, -1/2)$. The Jacobian of this map is

$$J = \begin{bmatrix} 0 & 1+\mu \\ -1-2y & 1-2x \end{bmatrix}$$

Consequently, the multipliers associated with the nontrivial fixed point are $\rho = 0$ and $\rho = 2 + \mu$ and hence it is unstable for values of μ near zero. On the other hand, the multipliers associated with the trivial fixed point are

$$\rho = \tfrac{1}{2} \pm \tfrac{1}{2} i\sqrt{3+4\mu} = \sqrt{1+\mu}\, e^{i\tan^{-1}\left(\sqrt{3}+\frac{2\sqrt{3}}{3}\mu\right)}$$

These complex-conjugate multipliers transversely exit the unit circle away from the real axis as μ increases past zero, and hence the trivial fixed point undergoes a Neimark–Sacker bifurcation as μ increases past zero.

To analyze the dynamics of the map near this bifurcation, we construct the normal form of the map (3.110) and (3.111) for small x, y, and μ. To this end, we rewrite it as

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix} + \begin{bmatrix} 0 & \mu \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix} + \begin{bmatrix} 0 \\ -2x_k y_k \end{bmatrix} \tag{3.112}$$

The eigenvalues of the linear part of (3.112) when $\mu = 0$ are

$$\rho_1 = \tfrac{1}{2}(1 + i\sqrt{3}) = e^{\frac{1}{3}i\pi} \quad \text{and} \quad \rho_2 = \bar{\rho}_1 = \tfrac{1}{2}(1 - i\sqrt{3}) = e^{-\frac{1}{3}i\pi}$$

and the associated eigenvectors are the columns of the matrix

$$P = \begin{bmatrix} \frac{1}{2}(1 - i\sqrt{3}) & \frac{1}{2}(1 + i\sqrt{3}) \\ 1 & 1 \end{bmatrix}$$

Next, we introduce the transformation

$$\begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} z \\ \bar{z} \end{bmatrix} \tag{3.113}$$

Hence,

$$x = \tfrac{1}{2}(1 - i\sqrt{3})z + \tfrac{1}{2}(1 + i\sqrt{3})\bar{z} \quad \text{and} \quad y = z + \bar{z} \tag{3.114}$$

Substituting (3.113) and (3.114) into (3.112) and multiplying the outcome from the left with P^{-1}, we obtain

$$\begin{aligned} z_{k+1} = {} & e^{\frac{1}{3}i\pi} z_k + \frac{1}{3}\sqrt{3}i\mu(z_k + \bar{z}_k) + \frac{2\sqrt{3}}{3} i z_k^2 \\ & - \left(1 - \frac{\sqrt{3}}{3}i\right) z_k \bar{z}_k - \left(1 + \frac{\sqrt{3}}{3}i\right) \bar{z}_k^2 \end{aligned} \tag{3.115}$$

with the second equation being the complex-conjugate of this equation.

Next, we determine the normal form of (3.115) for small z and μ. Instead of using two successive transformations to simplify the quadratic terms and then the cubic terms, we use a single transformation to accomplish simplification of both nonlinearities. To this end, we introduce a small nondimensional bookkeeping parameter ϵ, scale z as ϵz and μ as $\epsilon^2\mu$, let

$$z = \xi + \epsilon h_1(\xi, \bar{\xi}) + \epsilon^2 h_2(\xi, \bar{\xi}) \tag{3.116}$$

and choose $h_1(\xi, \bar{\xi})$ and $h_2(\xi, \bar{\xi})$ so that (3.115) takes on the simplest form

$$\xi_{k+1} = e^{\frac{1}{3}i\pi} \xi_k + \epsilon g_1(\xi_k, \bar{\xi}_k) + \epsilon^2 g_2(\xi_k, \bar{\xi}_k) \tag{3.117}$$

Substituting (3.116) and (3.117) into (3.115) and equating coefficients of like powers of ϵ, we obtain

Order (ϵ)

$$\begin{aligned} & g_1(\xi, \bar{\xi}) + h_1\left(e^{\frac{1}{3}i\pi}\xi, e^{-\frac{1}{3}i\pi}\bar{\xi}\right) - e^{\frac{1}{3}i\pi} h_1(\xi, \bar{\xi}) \\ & = \frac{2\sqrt{3}}{3} i\xi^2 - \left(1 - \frac{\sqrt{3}}{3}i\right)\xi\bar{\xi} - \left(1 + \frac{\sqrt{3}}{3}i\right)\bar{\xi}^2 \end{aligned} \tag{3.118}$$

Order (ϵ^2)

$$\begin{aligned} & g_2(\xi, \bar{\xi}) + h_2\left(e^{\frac{1}{3}i\pi}\xi, e^{-\frac{1}{3}i\pi}\bar{\xi}\right) - e^{\frac{1}{3}i\pi} h_2(\xi, \bar{\xi}) \\ & = \frac{1}{3}\sqrt{3}i\mu(\xi + \bar{\xi}) + \frac{4\sqrt{3}}{3} i\xi h_1(\xi, \bar{\xi}) \\ & \quad - e^{\frac{1}{3}i\pi} g_1(\xi, \bar{\xi}) \frac{\partial}{\partial \xi} h_1(\xi, \bar{\xi}) - \left(1 - \frac{\sqrt{3}}{3}i\right)\left(\xi \bar{h}_1(\xi, \bar{\xi}) + \bar{\xi} h_1(\xi, \bar{\xi})\right) \\ & \quad - e^{-\frac{1}{3}i\pi} \bar{g}_1(\xi, \bar{\xi}) \frac{\partial}{\partial \bar{\xi}} \bar{h}_1(\xi, \bar{\xi}) - 2\left(1 + \frac{\sqrt{3}}{3}i\right)\bar{\xi}\bar{h}_1(\xi, \bar{\xi}) \end{aligned} \tag{3.119}$$

Because $e^{1/3in\pi} \neq 1$ for $n = 1$ and 3, there are no strong resonances, and hence $h_1(\xi, \bar{\xi})$ can be chosen to eliminate all of the quadratic terms in (3.118); that is,

$$h_1(\xi, \bar{\xi}) = -\frac{2\sqrt{3}}{3} i \xi^2 - \left(1 + \frac{\sqrt{3}}{3} i\right) \xi \bar{\xi} + \frac{1}{2}\left(1 - \frac{\sqrt{3}}{3} i\right) \bar{\xi}^2 \tag{3.120}$$

Consequently, $g_1(\xi, \bar{\xi}) \equiv 0$ and (3.119) becomes

$$\begin{aligned} & g_2(\xi, \bar{\xi}) + h_2\left(e^{\frac{1}{3}i\pi}\xi, e^{-\frac{1}{3}i\pi}\bar{\xi}\right) - e^{\frac{1}{3}i\pi} h_2(\xi, \bar{\xi}) \\ & = \tfrac{1}{3}\sqrt{3} i \mu (\xi + \bar{\xi}) + 2\xi^3 + 2(1 - i\sqrt{3})\xi^2\bar{\xi} + 4\xi\bar{\xi}^2 + (1 - i\sqrt{3})\bar{\xi}^3 \end{aligned} \tag{3.121}$$

One can choose $h_2(\xi, \bar{\xi})$ to eliminate the nonresonance terms in (3.122), leaving $g_2(\xi, \bar{\xi})$ with the resonance terms; that is,

$$g_2(\xi, \bar{\xi}) = \tfrac{1}{3}\sqrt{3} i \mu \xi + 2(1 - i\sqrt{3})\xi^2\bar{\xi} \tag{3.122}$$

Therefore, the normal form of the map is

$$\xi_{k+1} = e^{\frac{1}{3}i\pi}\xi_k + \tfrac{1}{3}\sqrt{3} i \mu \xi_k + 2(1 - i\sqrt{3})\xi_k^2\bar{\xi}_k \tag{3.123}$$

where the bookkeeping parameter has been set equal to unity.

Equation 3.123 can be rewritten as

$$\xi_{k+1} = \left(1 + \tfrac{1}{2}\mu\right) e^{\frac{1}{3}i\pi + \frac{1}{6}\sqrt{3}i\mu}\left[\xi_k - (2 + 2i\sqrt{3})\xi_k^2\bar{\xi}_k\right] + \cdots \tag{3.124}$$

Letting $\xi = re^{i\theta}$ in (3.124), we obtain

$$r_{k+1} = \left(1 + \tfrac{1}{2}\mu\right) r_k - 2r_k^3 \tag{3.125}$$

$$\theta_{k+1} = \theta_k + \tfrac{1}{3}\pi + \tfrac{1}{6}\sqrt{3}\mu - 2\sqrt{3}r_k^2 \tag{3.126}$$

It follows from (3.125) that the attracting invariant circle $r = 1/2\sqrt{\mu}$ bifurcates from the origin as μ increases past zero. The dynamics on this invariant circle is periodic or quasiperiodic, depending on whether $1/3(\pi - \sqrt{3}\mu)$ is rational or irrational. We note that higher-order terms in μ and r have been neglected in arriving at (3.124) and (3.125). Including these higher-order terms, one finds that the attracting smooth invariant curve is a circle only for values of μ close to zero. Moreover, for large values of μ, the attractor may be complicated. In Figure 3.8, we show phase portraits obtained numerically for the map (3.110) and (3.111) for $\mu = -0.01$ and $\mu = 0.02$. When $\mu < 0$, the origin is asymptotically stable and all iterates starting within its domain of attraction spiral to it, as shown in Figure 3.8a. The six-fold rotational symmetry is the result of β being $1/3\pi$. When $\mu > 0$, the origin is unstable and all iterates starting close to the origin spiral out to a smooth closed invariant curve enclosing the origin, as shown in Figure 3.8b. For the chosen value of μ, the long-time iterates densely fill the invariant curve. Again, the six-fold rotational symmetry is the result of β being $1/3\pi$.

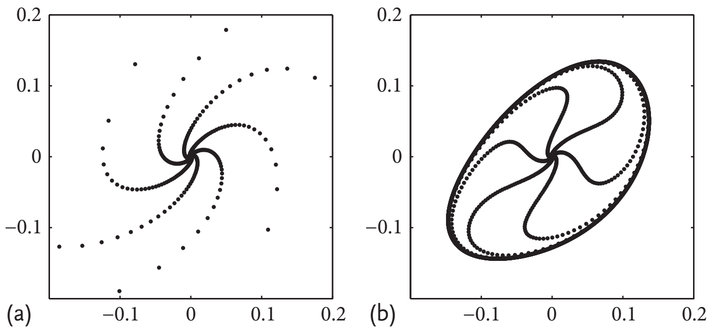

Figure 3.8 Phase portrait of the map (3.110) and (3.111): (a) $\mu < 0$ and the origin is stable and iterates spiral to it and (b) $\mu > 0$ and the origin is unstable and iterates spiral away from it and onto a smooth invariant closed curve encircling it.

3.5 Exercises

3.5.1 Consider the map

$$x_{k+1} = x_k + \mu - x_k^2$$

Examine the bifurcation that occurs at $(x_k, \mu) = (1, 1)$.

3.5.2 Consider the map

$$x_{k+1} = x_k + \mu x_k - x_k^2$$

Study the bifurcation that takes place at $(x, \mu) = (0, -2)$.

3.5.3 Consider the following map:

$$x_{k+1} = \mu + x_k^2$$

Examine the bifurcation that takes place at $(x, \mu) = (1/2, 1/4)$.

3.5.4 Consider the following map:

$$x_{k+1} = \mu x_k + x_k^2$$

Examine the bifurcation that takes place at $(x, \mu) = (0, 1)$.

3.5.5 Consider the map

$$x_{k+1} = -a x_k^2 + 1$$

Determine its fixed points and their stability. Examine the bifurcation that takes place as a is varied.

3.5.6 Find the fixed points of

$$x_{n+1} = a x_n - x_n^3$$

Determine their stability and the bifurcations, which they undergo as a is varied.

3.5.7 Find the fixed points of

$$x_{n+1} = a x_n + x_n^3$$

Determine their stability and the bifurcations, which they undergo as a is varied.

3.5.8 Consider the following map:

$$x_{k+1} = \mu - x_k + x_k^2$$

Determine whether the period-doubled orbit bifurcating from the point $(x, \mu) = (0, 0)$ is supercritical or subcritical.

3.5.9 Consider the map

$$x_{n+1} = -(1 + a)x_n + b x_n^2 - x_n^3$$

What is the type of bifurcation which occurs at $x = 0$ and $a = 0$? Is it supercritical or subcritical?

3.5.10 Show that a fixed point of the map $f(x; \lambda) = (1 + \lambda)x + x^2$ undergoes a transcritical bifurcation at $\lambda = 0$.

3.5.11 Show that the map $f(x; \lambda) = e^x - \lambda$ undergoes a saddle-node bifurcation at $\lambda = 1$.

3.5.12 Determine the fixed points of the following cubic map and discuss their stability:

$$x_{n+1} = (1 - \lambda)x_n + \lambda x_n^3$$

For what value of λ does the first period-doubling bifurcation occur?

3.5.13 Consider the map

$$x_{k+1} = -x_k + a x_k^2 + b x_k^3$$

Compute the second iterate of this map and hence determine whether the period-doubled orbit is stable or unstable.

3.5.14 Determine the normal form of the map

$$x_{k+1} = x_k + a x_k^2 + b x_k^3$$

3.5.15 Consider the two one-dimensional maps

$$x_{n+1} = e^{x_n} - \lambda$$
$$y_{n+1} = -\tfrac{1}{2}\lambda \tan^{-1} y_n$$

Find the fixed points and their stability. Show that x undergoes a saddle-node bifurcation at $\lambda = 1$, whereas y undergoes a period-doubling bifurcation at $\lambda = 3$.

3.5.16 Consider the Hénon map

$$x_{k+1} = 1 + y_k - \alpha x_k^2$$
$$y_{k+1} = \beta x_k$$

Assume that $\beta \neq 0$ and $\alpha > 0$.

a) Verify that this map has a stable fixed point and an unstable fixed point when

$$\alpha < \tfrac{3}{4}(1-\beta)^2$$

b) Is this map dissipative for $\beta = 0.3$? For this case, determine the period-one and period-two points of this map for $\alpha = 0.1$, $\alpha = 0.5$, and $\alpha = 1.3$. Discuss their stability.
c) For the above values of α, plot the iterates of this map.
d) Examine the bifurcation that takes place at

$$(x_k, y_k, \alpha) = \left[\frac{(1-\beta)}{2\alpha}, \frac{\beta(1-\beta)}{2\alpha}, \frac{3}{4}(1-\beta)^2\right].$$

3.5.17 Consider the two-dimensional map

$$x_{n+1} = \lambda + x_n + \lambda y_n + x_n^2$$
$$y_{n+1} = \tfrac{1}{2} y_n + \lambda x_n + x_n^2$$

Determine its fixed points and then show that the origin undergoes a saddle-node bifurcation at $\lambda = 0$.

3.5.18 Consider the two-dimensional map:

$$x_{n+1} = y_n$$
$$y_{n+1} = -\tfrac{1}{2} x_n + \lambda y_n - y_n^3$$

Determine its fixed points. Show that the origin undergoes a pitchfork bifurcation at $\lambda = 3/2$. Analyze the bifurcation at $\lambda = 3$.

3.5.19 Consider the following map:

$$x_{k+1} = 2x_k$$
$$y_{k+1} = \tfrac{1}{2} y_k + 7x_k^2$$

Show that the stable manifold of its fixed point is the y-axis and the unstable manifold is $y = 2x^2$.

3.5.20 Consider the map

$$x_{k+1} = y_k$$
$$y_{k+1} = \alpha y_k(1 - x_k)$$

Show that the origin is a saddle. Determine analytical expressions for the stable and unstable manifolds.

3.5.21 Consider the map

$$x_{k+1} = y_k$$
$$y_{k+1} = \alpha y_k(1 - x_k)$$

Examine the bifurcation that occurs at $\alpha = 2$ and determine the normal form of the map near this bifurcation.

3.5.22 Consider the map

$$x_{k+1} = y_k$$
$$y_{k+1} = \alpha y_k(1 - x_k)$$

Examine the bifurcation that occurs at $\alpha = 1$ and determine the normal form of the map near this bifurcation.

3.5.23 Consider the following map near the origin:

$$x_{k+1} = x_k + \alpha x_k y_k$$
$$y_{k+1} = \tfrac{1}{2} y_k + x_k^2$$

Compute the center manifold, describe the dynamics on the center manifold, and then determine the stability of the origin.

3.5.24 Consider the map

$$x_{n+1} = y_n$$
$$y_{n+1} = b x_n - y_n + x_n y_n$$

What type of bifurcation occurs at $(x, y, b) = (0, 0, 0)$.

3.5.25 Consider the map

$$x_{k+1} = \tfrac{5}{8} x_k + \tfrac{3}{8} y_k + xy$$
$$y_{k+1} = \tfrac{3}{8} x_k + \tfrac{5}{8} y_k + x^2 - y^2$$

Compute the center manifold near the origin, describe the dynamics on the center manifold, and then determine the stability of the origin.

3.5.26 Consider the two-dimensional map

$$x_{n+1} = (1+\mu)y_n$$
$$y_{n+1} = y_n - x_n - 2x_n y_n$$

Determine the fixed points of this map and their stability. Show that the origin undergoes a Hopf bifurcation at $\mu = 0$. Calculate and plot the phase portraits for

a) $\mu = -0.01$ using the initial condition $x = 0.4$, $y = -0.4$.
b) $\mu = 0.05$ using the initial condition $x = 0.01$, $y = -0.01$.

3.5.27 Consider the map

$$x_{k+1} = y_k$$
$$y_{k+1} = \tfrac{1}{2}\mu + (1+\mu)(y_k - x_k - 2x_k y_k)$$

a) Determine the fixed points and their stability.
b) Compute the normal form of this map near the origin when $\mu \approx 0$.
c) Use this normal form to calculate the bifurcating invariant circle from the origin.
d) Calculate the phase portraits for $\mu = -0.01$ and $\mu = 0.02$.

3.5.28 Consider the map

$$z_{k+1} = \rho z_k + b_{11}z^2 + b_{12}z\bar{z} + b_{13}\bar{z}^2 + a_{11}z^3 + a_{12}z^2\bar{z} + a_{13}z\bar{z}^2 + a_{14}\bar{z}^3$$

Show that its normal form near the origin has the form

$$z_{k+1} = \rho z_k + \alpha_1\bar{z}^2 + \alpha_2 z^2\bar{z}$$

when $\rho^3 = 1$ and the form

$$z_{k+1} = \rho z_k + \alpha_1 z^2\bar{z} + \alpha_2\bar{z}^3$$

when $\rho^4 = 1$.

3.5.29 Consider the map

$$z_{n+1} = \tfrac{1}{2}(\sqrt{3} + i + 2i\mu)z_n + z_n^2\bar{z}_n$$

What type of bifurcation occurs at $(z, \bar{z}, \mu) = (0, 0, 0)$. Determine the bifurcating solutions.

4
Bifurcations of Continuous Systems

In this chapter, we construct normal forms of smooth continuous systems depending on a scalar control parameter μ near their fixed or equilibrium points. Specifically, we consider the following system:

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}; \mu)$$

where $\boldsymbol{x} \in U \subset \mathcal{R}^n$, $\boldsymbol{F} \in U \subset \mathcal{R}^n$, and $\mu \in V \subset \mathcal{R}$. The *fixed points* or *equilibrium solutions* of this system are solutions of the algebraic system of equations

$$\boldsymbol{F}(\boldsymbol{x}; \mu) = \boldsymbol{0}$$

In Section 4.1, we consider linear systems; in Section 4.2, we consider the fixed points of nonlinear systems and their stability; in Section 4.3, we discuss center-manifold reduction; in Section 4.4, we consider local bifurcations of fixed points; and in Sections 4.5 and 4.6, we illustrate how the method of multiple scales, a combination of center-manifold reduction and the method of normal forms, and a projection method can be used to construct the normal forms of static and Hopf bifurcations in the neighborhood of a fixed point.

4.1
Linear Systems

We consider the linear system

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} \tag{4.1}$$

where A is an $n \times n$ constant matrix. In this case, the trivial solution $\boldsymbol{x}^* = \boldsymbol{0}$ is a fixed point of this linear system. We denote the eigenvalues of A by $\lambda_i, i = 1, 2, \ldots, n$, and the corresponding eigenvectors (generalized eigenvectors) by $\boldsymbol{p}_i$, $i = 1, 2, \ldots, n$. The eigenvalues are the roots of the characteristic equation

$$\det(A - \lambda I) = 0 \tag{4.2}$$

The eigenvector $\boldsymbol{p}_i$ corresponding to a distinct eigenvalue λ_i is given by

$$A\boldsymbol{p}_i = \lambda_i \boldsymbol{p}_i \tag{4.3}$$

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and the generalized eigenvectors corresponding to an eigenvalue λ_m with multiplicity n_m are the nontrivial solutions of

$$(A - \lambda_m \boldsymbol{I})\boldsymbol{p} = \boldsymbol{0}\,, \quad (A - \lambda_m \boldsymbol{I})^2\boldsymbol{p} = \boldsymbol{0}\,, \quad \ldots\,, \quad (A - \lambda_m \boldsymbol{I})^{n_m}\boldsymbol{p} = \boldsymbol{0} \tag{4.4}$$

If an eigenvalue is complex-valued, then its corresponding eigenvector and generalized eigenvectors are also complex-valued.

Introducing the transformation

$$\boldsymbol{x} = P\boldsymbol{y} \tag{4.5}$$

where the matrix $P = [\boldsymbol{p}_1 \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n]$, into (4.1) yields

$$P\dot{\boldsymbol{y}} = AP\boldsymbol{y} \tag{4.6}$$

Multiplying (4.6) from the left with the inverse P^{-1} of P, we obtain the normal of (4.1) as

$$\dot{\boldsymbol{y}} = J\boldsymbol{y} \tag{4.7}$$

where $J = P^{-1}AP$ is called the Jordan canonical form of A. Next, we discuss two cases: systems with distinct and nondistinct eigenvalues.

4.1.1 Case of Distinct Eigenvalues

If the eigenvalues of A are distinct, then J is a diagonal matrix D with entries λ_i, $i = 1, 2, \ldots, n$; that is,

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \lambda_2 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \lambda_n \end{bmatrix} \tag{4.8}$$

Then, (4.7) can be rewritten as

$$\dot{y}^{(m)} = \lambda_m y^{(m)}\,, \quad m = 1, 2, \ldots, n \tag{4.9}$$

where $y^{(m)}$ is the mth component of $\boldsymbol{y}$. Therefore,

$$y^{(m)} = c_m e^{\lambda_m t}\,, \quad m = 1, 2, \ldots, n \tag{4.10}$$

where c_m is a constant. It follows from (4.10) that $y^{(m)} \to 0$ as $t \to \infty$ when λ_m lies in the left-half of the complex plane, $y^{(m)} \to \infty$ as $t \to \infty$ when λ_m lies in the right-half of the complex plane, and $y^{(m)} = c_m$ for all time when λ_m lies on the imaginary axis. Therefore, the origin of (4.1) is (a) asymptotically stable if all of the eigenvalues λ_i of the matrix A lie in the left-half of the complex plane, (b) unstable if one or more λ_i lie in the right-half of the complex plane, and (c) neutrally or marginally stable if one or more λ_i lie on the imaginary axis with the rest of the eigenvalues being in the left-half of the complex plane.

4.1.2 Case of Repeated Eigenvalues

If the number of distinct eigenvalues of A is $k < n$, then J is diagonal if all of the p_i are eigenvectors; otherwise, it has the form

$$J = \begin{bmatrix} J_1 & \phi & \cdot & \cdot & \cdot & \phi \\ \phi & J_2 & \cdot & \cdot & \cdot & \phi \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \phi & \phi & \cdot & \cdot & \cdot & J_k \end{bmatrix} \tag{4.11}$$

where ϕ represents a matrix with zero entries and

$$J_m = \begin{bmatrix} \lambda_m & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \lambda_m & 1 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \lambda_m & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \lambda_m \end{bmatrix} \tag{4.12}$$

Example 4.1

We consider a system with repeated roots and a nondiagonal J; that is,

$$A = \begin{bmatrix} a & \frac{1}{2}(a-b) \\ -\frac{1}{2}(a-b) & b \end{bmatrix}$$

where $b \neq a$. The eigenvalues of this matrix are $\rho = 1/2(a+b)$ with a multiplicity of two. It follows from Example 3.3. that the corresponding eigenvector and generalized eigenvector are

$$\boldsymbol{p}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and

$$\boldsymbol{p}_2 = \begin{bmatrix} -1 \\ 1 + \dfrac{2}{b-a} \end{bmatrix}$$

Then,

$$P^{-1}AP = J = \begin{bmatrix} \dfrac{1}{2}(a+b) & 1 \\ 0 & \dfrac{1}{2}(a+b) \end{bmatrix}$$

and (4.7) yields

$$\dot{y}^{(1)} = \lambda y^{(1)} + y^{(2)} \tag{4.13}$$

$$\dot{y}^{(2)} = \lambda y^{(2)} \tag{4.14}$$

The general solutions of (4.13) and (4.14) can be expressed as

$$y^{(1)} = c_1 e^{\lambda t} + t c_2 e^{\lambda t} \tag{4.15}$$

$$y^{(2)} = c_2 e^{\lambda t} \tag{4.16}$$

where c_1 and c_2 are arbitrary constants. Therefore, the origin of (4.1) is (a) asymptotically stable if Real(λ) < 0 and (b) unstable if Real(λ) ≥ 0.

4.2
Fixed Points of Nonlinear Systems

The fixed points of the autonomous system

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}; \mu) \tag{4.17}$$

are defined by the vanishing of the vector field; that is,

$$\boldsymbol{F}(\boldsymbol{x}; \mu) = \boldsymbol{0} \tag{4.18}$$

A location in the state space where this condition is satisfied is called a *singular point*. At such a point, the integral curve of the vector field $\boldsymbol{F}$ corresponds to the point itself. Also, an orbit of a fixed point is the fixed point itself. Fixed points are also called *stationary solutions, critical points, constant solutions,* and sometimes *steady-state solutions*. Physically, *a fixed point corresponds to an equilibrium position of a system*. Further, fixed points are examples of invariant sets of (4.17).

4.2.1
Stability of Fixed Points

To investigate the stability of a fixed point $\boldsymbol{x}^*(\mu^*)$, where $\boldsymbol{x}^* \in \mathcal{R}^n$ and $\mu^* \in \mathcal{R}$, we superimpose on it a small disturbance $\boldsymbol{y}(t)$ and obtain

$$\boldsymbol{x}(t) = \boldsymbol{x}^* + \boldsymbol{y}(t) \tag{4.19}$$

Substituting (4.19) into (4.17) yields

$$\dot{\boldsymbol{y}} = \boldsymbol{F}(\boldsymbol{x}^* + \boldsymbol{y}; \mu^*) \tag{4.20}$$

We note that the fixed point $\boldsymbol{x} = \boldsymbol{x}^*$ of (4.17) has been transformed into the fixed point $\boldsymbol{y} = \boldsymbol{0}$ of (4.20). Assuming that $\boldsymbol{F}$ is at least twice continuously differentiable (i.e., $\mathcal{C}^2$), expanding (4.20) in a Taylor series about $\boldsymbol{x}^*$, and retaining only linear terms in the disturbance leads to

$$\dot{\boldsymbol{y}} = \boldsymbol{F}(\boldsymbol{x}^*; \mu^*) + D_{\boldsymbol{x}} \boldsymbol{F}(\boldsymbol{x}^*; \mu^*) \boldsymbol{y} + O(||\boldsymbol{y}||^2)$$

or

$$\dot{\boldsymbol{y}} \approx D_{\boldsymbol{x}}\boldsymbol{F}(\boldsymbol{x}^*;\mu^*)\boldsymbol{y} \equiv A\boldsymbol{y} \tag{4.21}$$

where A, the matrix of first partial derivatives, is called the *Jacobian matrix*. If the components of F are

$$F_1(x_1, x_2, \ldots, x_n),\ F_2(x_1, x_2, \ldots, x_n), \ldots,\ F_n(x_1, x_2, \ldots, x_n)$$

then

$$A = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdot & \cdot & \cdot & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdot & \cdot & \cdot & \frac{\partial F_2}{\partial x_n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdot & \cdot & \cdot & \frac{\partial F_n}{\partial x_n} \end{bmatrix}$$

We have transformed the problem of determining the local stability of the fixed point $\boldsymbol{x}^*$ of (4.17) into that of determining the stability of the trivial solution of the linear system (4.21). We say *local stability* because we have considered a small disturbance and linearized the vector field. It follows from the preceding section that the trivial solution of (4.21) and hence the fixed point $\boldsymbol{x}^*$ of (4.17) is (a) asymptotically stable if all of the eigenvalues λ_i of the matrix A lie in the left-half of the complex plane and (b) unstable if one or more λ_i lie in the right-half of the complex plane. In the case of repeated eigenvalues, the trivial solution of (4.21) and hence the fixed point $\boldsymbol{x}^*$ of (4.17) is unstable if one or more eigenvalues lie on the imaginary axis and the Jordan form is not diagonal.

4.2.2
Classification of Fixed Points

When all of the eigenvalues of A have nonzero real parts, the corresponding fixed point is called a *hyperbolic fixed point*, irrespective of the values of the imaginary parts; otherwise, it is called a *nonhyperbolic fixed point*.

There are three types of hyperbolic fixed points: *sinks*, *sources*, and *saddles*. If all of the eigenvalues of A have negative real parts, then all of the components of the disturbance $\boldsymbol{y}$ decay in time, and hence $\boldsymbol{x}$ approaches the fixed point $\boldsymbol{x}^*$ of (4.17) as $t \to \infty$. Therefore, the fixed point $\boldsymbol{x}^*$ of (4.17) is asymptotically stable. An asymptotically stable fixed point is called a *sink*. If the matrix A associated with a sink has complex eigenvalues, the sink is also called a *stable focus*. On the other hand, if all of the eigenvalues of the matrix A associated with a sink are real, the sink is also called a *stable node*. A sink is stable in forward time (i.e., $t \to \infty$) but unstable in reverse time (i.e., $t \to -\infty$). Further, all sinks qualify as attractors.

If one or more of the eigenvalues of A have positive real parts, some of the components of $\boldsymbol{y}$ grow in time, and $\boldsymbol{x}$ moves away from the fixed point $\boldsymbol{x}^*$ of (4.17) as

t increases. In this case, the fixed point $\boldsymbol{x}^*$ is said to be unstable. When all of the eigenvalues of A have positive real parts, $\boldsymbol{x}^*$ is said to be a *source*. If the matrix A associated with a source has complex eigenvalues, the source is also called an *unstable focus*. On the other hand, if all of the eigenvalues of the matrix A associated with a source are real, the source is also called an *unstable node*. A source is unstable in forward time but stable in reverse time. Because trajectories move away from a source in forward time, the source is an example of a repellor.

When some, but not all, of the eigenvalues have positive real parts while the rest of the eigenvalues have negative real parts, the associated fixed point is called a *saddle point*. Because a saddle point is unstable in both forward and reverse times, it is called a *nonstable fixed point*.

A nonhyperbolic fixed point is unstable if one or more of the eigenvalues of A have positive real parts. If some of the eigenvalues of A have negative real parts while the rest of the eigenvalues are distinct and have zero real parts, the fixed point $\boldsymbol{x} = \boldsymbol{x}^*$ of (4.17) is said to be *neutrally* or *marginally stable*. If all of the eigenvalues of A are distinct, nonzero, and purely imaginary, the corresponding fixed point is called a *center*.

Example 4.2

We consider the system

$$\dot{x} = x(3 - x - 2y) \tag{4.22}$$

$$\dot{y} = y(2 - x - y) \tag{4.23}$$

Its fixed points are $(0,0)$, $(0,2)$, $(3,0)$, and $(1,1)$. The Jacobian matrix of the system (4.22) and (4.23) is

$$A = \begin{bmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{bmatrix}$$

The eigenvalues of A corresponding to the fixed point $(0,0)$ are $\lambda_1 = 2$ and $\lambda_2 = 3$; hence, it is an unstable node. The eigenvalues of A corresponding to the fixed point $(0,2)$ are $\lambda_1 = -1$ and $\lambda_2 = -2$; hence, it is a stable node. The eigenvalues of A corresponding to the fixed point $(3,0)$ are $\lambda_1 = -1$ and $\lambda_2 = -3$; hence, it is a stable node. The eigenvalues of A corresponding to the fixed point $(1,1)$ are $\lambda_1 = -1 + \sqrt{2}$ and $\lambda_2 = -1 - \sqrt{2}$; hence, it is a saddle. In this example, all of the fixed points are hyperbolic.

4.2.3
Hartman–Grobman and Shoshitaishvili Theorems

Many theorems provide precise statements on what the stability of fixed-point solutions of the linearized system (4.21) imply for the stability of fixed-point solutions of the full nonlinear system (4.17). The *Hartman–Grobman theorem* (e.g., Arnold, 1988, Chapter 3; Wiggins, 1990, Chapter 2) is applicable to hyperbolic fixed points, whereas the *Shoshitaishvili theorem* (e.g., Arnold, 1988, Chapter 6) is applicable to nonhyperbolic fixed points. From these theorems, it follows that (a) the fixed point $\boldsymbol{x} = \boldsymbol{x}^*$ of the nonlinear system (4.17) is stable when the fixed point $\boldsymbol{y} = \mathbf{0}$ of the linear system (4.21) is asymptotically stable; (b) the fixed point $\boldsymbol{x} = \boldsymbol{x}^*$ of the nonlinear system (4.17) is unstable when the fixed point $\boldsymbol{y} = \mathbf{0}$ of the linear system (4.21) is unstable; and (c) linearization cannot determine the stability of neutrally stable fixed points (including centers) of (4.17). In the case of neutrally stable fixed points, a nonlinear analysis is necessary to determine the stability of $\boldsymbol{x}^*$. It will be necessary to retain quadratic and, sometimes, higher-order terms in the disturbance $\boldsymbol{y}$ in the Taylor-series expansion of (4.20).

In a topological setting, the Hartman–Grobman theorem implies that the trajectories in the vicinity of a hyperbolic fixed point $\boldsymbol{x} = \boldsymbol{x}^*$ of (4.17) are qualitatively similar to those in the vicinity of the hyperbolic fixed point $\boldsymbol{y} = \mathbf{0}$ of (4.21). In other words, the local nonlinear dynamics near $\boldsymbol{x} = \boldsymbol{x}^*$ is qualitatively similar to the linear dynamics near $\boldsymbol{y} = \mathbf{0}$, and a qualitative change in the local nonlinear dynamics can be detected by examining the associated linear dynamics.

According to the Hartman–Grobman theorem, there exists a continuous coordinate transformation (i.e., a homeomorphism) that transforms the nonlinear flow into the linear flow in the vicinity of a hyperbolic fixed point. In the absence of resonances or near resonances, the method of normal forms (e.g., Sections 2.2 and 2.4) may be used to generate a coordinate transformation to transform the nonlinear flow into linear flow (e.g., Arnold, 1988; Guckenheimer and Holmes, 1983). Further, such a coordinate transformation would be a differentiable one because the method of normal forms yields transformations in the form of power-series expansions.

4.3 Center-Manifold Reduction

We consider the local dynamics near a nonhyperbolic fixed point $\boldsymbol{x}^*$ of the nonlinear system (4.17), where $\boldsymbol{F}$ is an analytic vector function of $\boldsymbol{x}$. According to the center-manifold theorem (Carr, 1981), there exists a C^r local center manifold for the nonlinear system (4.17) near $\boldsymbol{x}^*$. Furthermore, if none of the eigenvalues of this fixed point lies in the right-half of the complex plane, the long-time dynamics of (4.17) can be reduced to determining the dynamics on the center manifold. Next, we describe how to construct the center manifold in the neighborhood of a nonhyperbolic fixed point with one eigenvalue being zero and the real parts of all of the other eigenvalues being negative. We assume that the fixed point has been shifted to the origin and that the linear part has been transformed into a Jordan canonical form; that is, we consider the system

$$\dot{\boldsymbol{x}}_{k+1} = J\boldsymbol{x}_k + \boldsymbol{F}(\boldsymbol{x}) \tag{4.24}$$

We arrange the system (4.24) and rewrite it as

$$\dot{x} = f(x, \boldsymbol{y}) \tag{4.25}$$

$$\dot{\boldsymbol{y}} = B\boldsymbol{y} + \boldsymbol{G}(x, \boldsymbol{y}) \tag{4.26}$$

where B is a constant matrix with the real parts of all of its eigenvalues being negative and f and $\boldsymbol{G}$ are scalar and vector-valued nonlinear functions of x and $\boldsymbol{y}$. According to the center-manifold theorem, there exists a center manifold

$$\boldsymbol{y} = \boldsymbol{h}(x) \tag{4.27}$$

Moreover, the dynamics of the system (4.25) and (4.26) is qualitatively similar to the dynamics on this manifold; that is,

$$\dot{x} = f\left[x, \boldsymbol{h}(x)\right] \tag{4.28}$$

Substituting (4.27) into (4.26) yields

$$\dot{\boldsymbol{h}}(x) = B\boldsymbol{h}(x) + \boldsymbol{G}\left[x, \boldsymbol{h}(x)\right] \tag{4.29}$$

which upon using (4.28) becomes

$$\boldsymbol{h}'(x) f\left[x, \boldsymbol{h}(x)\right] = B\boldsymbol{h}(x) + \boldsymbol{G}\left[x, \boldsymbol{h}(x)\right] \tag{4.30}$$

Using three examples, we describe how to construct approximate solutions of (4.30).

Example 4.3

We consider the system

$$\dot{x} = \alpha_1 x y \tag{4.31}$$

$$\dot{y} = -y + \alpha_2 x^2 \tag{4.32}$$

where α_1 and α_2 are constants. Clearly, the origin is a fixed point of (4.31) and (4.32). Because its associated eigenvalues are $\lambda = 0$ and $\lambda = -1$, the origin is nonhyperbolic and the center and stable subspaces are one-dimensional. Consequently, the center manifold is one-dimensional. To calculate the center manifold and the dynamics on this manifold, we note that $B = -1$, $f(x, y) = \alpha_1 x y$, and $G(x, y) = \alpha_2 x^2$. Hence, (4.30) becomes

$$h'(x)\alpha_1 x h(x) = -h(x) + \alpha_2 x^2 \tag{4.33}$$

We seek an approximate solution of (4.33) in the form

$$h(x) = ax^2 + \cdots \tag{4.34}$$

and obtain

$$2a^2\alpha_1 x^4 + ax^2 - \alpha_2 x^2 + \cdots = 0 \tag{4.35}$$

Equating to zero the coefficient of x^2, we obtain $a = \alpha_2$. Hence, the center manifold is given by

$$h(x) = \alpha_2 x^2 + \cdots \tag{4.36}$$

and it follows from (4.31) that the long-time dynamics on this center manifold is given by

$$\dot{x} = \alpha_1 \alpha_2 x^3 + \cdots \tag{4.37}$$

We note that linearization is not sufficient for determining the stability of the origin because its associated eigenvalues are $\lambda = 0$ and $\lambda = -1$. However, including the nonlinear terms, we find from (4.37) that the origin is unstable when $\alpha_1\alpha_2 > 0$ and stable when $\alpha_1\alpha_2 < 0$.

Example 4.4

We consider the system

$$\dot{x} = \alpha_1 x y + \alpha_2 x z \tag{4.38}$$

$$\dot{y} = -\mu y + z + \alpha_3 x^2 \tag{4.39}$$

$$\dot{z} = -\mu z - y + \alpha_4 x^2 \tag{4.40}$$

where $\mu > 0$. Clearly, the origin is a fixed point of (4.38)–(4.40) and its associated eigenvalues are $\lambda = 0$ and $\lambda = -\mu \pm i$. Hence, the origin is a nonhyperbolic fixed point and the center and stable subspaces are one-dimensional and two-dimensional, respectively. Consequently, the center manifold is two-dimensional. Thus, we seek the center manifold of (4.38)–(4.40) emanating from the origin in the form

$$y(x) = ax^2 + \cdots \quad \text{and} \quad z(x) = bx^2 + \cdots \tag{4.41}$$

Substituting (4.41) into (4.38) leads to the following equation describing the dynamics on the manifold:

$$\dot{x} = \alpha_1 a x^3 + \alpha_2 b x^3 + \cdots \tag{4.42}$$

Substituting (4.41) and (4.42) into (4.39) and (4.40) yields

$$2ax(\alpha_1 a x^3 + \alpha_2 b x^3) + \mu a x^2 - b x^2 - \alpha_3 x^2 + \cdots = 0 \tag{4.43}$$

$$2bx(\alpha_1 a x^3 + \alpha_2 b x^3) + \mu b x^2 + a x^2 - \alpha_4 x^2 + \cdots = 0 \tag{4.44}$$

Equating the coefficients of x^2 to zero in (4.43) and (4.44), we obtain

$$\mu a - b = \alpha_3 \tag{4.45}$$

$$a + \mu b = \alpha_4 \tag{4.46}$$

Hence,

$$a = \frac{\mu\alpha_3 + \alpha_4}{1+\mu^2} \quad \text{and} \quad b = \frac{\mu\alpha_4 - \alpha_3}{1+\mu^2} \tag{4.47}$$

and it follows from (4.42) that the dynamics on the center manifold is given by

$$\dot{x} = \Gamma x^3 + \cdots \tag{4.48}$$

where

$$\Gamma = \frac{\mu(\alpha_1\alpha_3 + \alpha_2\alpha_4) + \alpha_1\alpha_4 - \alpha_2\alpha_3}{1+\mu^2}$$

We note again that linearization is not sufficient for determining the stability of the origin because its associated eigenvalues are $\lambda = 0$ and $\lambda = -\mu \pm i$. However, including the nonlinear terms, we find from (4.48) that the origin is unstable when $\Gamma > 0$ and stable when $\Gamma < 0$.

Example 4.5

We consider the system

$$\dot{x} = y + a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5xz + a_6yz \tag{4.49}$$

$$\dot{y} = -x + b_1x^2 + b_2y^2 + b_3z^2 + b_4xy + b_5xz + b_6yz \tag{4.50}$$

$$\dot{z} = -z + c_1x^2 + c_2y^2 + c_3z^2 + c_4xy + c_5xz + c_6yz \tag{4.51}$$

Clearly, the origin $x = y = z = 0$ is a fixed point of (4.49)–(4.51). Its associated eigenvalues are $\lambda = i, -i, -1$, and hence the center subspace is two-dimensional and the stable subspace is one-dimensional. Consequently, the center manifold is one-dimensional.

We seek the center manifold emanating from the origin in the form

$$z(x, y) = \alpha_1x^2 + \alpha_2xy + \alpha_3y^2 \tag{4.52}$$

Substituting (4.52) into (4.49) and (4.50) yields the following two-dimensional system describing the dynamics on the center manifold:

$$\begin{aligned}\dot{x} = {} & y + a_1x^2 + a_2y^2 + a_4xy \\ & + (a_5x + a_6y)(\alpha_1x^2 + \alpha_2xy + \alpha_3y^2) + \cdots\end{aligned} \tag{4.53}$$

$$\begin{aligned}\dot{y} = {} & -x + b_1x^2 + b_2y^2 + b_4xy \\ & + (b_5x + b_6y)(\alpha_1x^2 + \alpha_2xy + \alpha_3y^2) + \cdots\end{aligned} \tag{4.54}$$

Substituting (4.52) into (4.51) and using (4.53) and (4.54), we obtain

$$2\alpha_1 xy + \alpha_2 y^2 - \alpha_2 x^2 - 2\alpha_3 xy = -\alpha_1 x^2 - \alpha_2 xy - \alpha_3 y^2 + c_1 x^2 + c_2 y^2 + c_4 xy + \cdots \quad (4.55)$$

Equating the coefficients of x^2, xy, and y^2 on both sides of (4.55) yields

$$\alpha_1 - \alpha_2 = c_1 \quad (4.56)$$

$$2\alpha_1 - 2\alpha_3 + \alpha_2 = c_4 \quad (4.57)$$

$$\alpha_2 + \alpha_3 = c_2 \quad (4.58)$$

Therefore,

$$\alpha_1 = \tfrac{1}{5}(3c_1 + 2c_2 + c_4) \quad (4.59)$$

$$\alpha_2 = -\tfrac{1}{5}(2c_1 - 2c_2 - c_4) \quad (4.60)$$

$$\alpha_3 = \tfrac{1}{5}(2c_1 + 3c_2 - c_4) \quad (4.61)$$

Substituting for the α_i from (4.59)–(4.61) into (4.53) and (4.54), we obtain a two-dimensional dynamical system describing the dynamics on the center manifold. Using a methodology similar to that in Section 2.5, we reduce this dynamical system into its normal form.

4.4 Local Bifurcations of Fixed Points

From Section 4.2, we know that the matrix A in (4.21) and the associated eigenvalues are functions of the control parameter μ. Let us suppose that, as μ is slowly varied, a fixed point becomes nonhyperbolic at a certain location in the state-control space. Then, if the state-space portraits before and after this location are qualitatively different, this location is called a *bifurcation point*, and the accompanying qualitative change is called a *bifurcation*.

There are two cases in which a fixed point x^* of the continuous system (4.17) ceases to be hyperbolic at a critical value μ_c of the control parameter. These cases are

1. $D_x \boldsymbol{F}(\boldsymbol{x}^*; \mu_c)$ has one eigenvalue equal to zero with the remaining eigenvalues being in the left-half of the complex plane,
2. $D_x \boldsymbol{F}(\boldsymbol{x}^*; \mu_c)$ has a pair of purely imaginary eigenvalues with the remaining eigenvalues being in the left-half of the complex plane.

According to the center-manifold theorem, analysis of the the dynamics of an n-dimensional continuous system near one of its fixed points can be reduced to the analysis of the dynamics on its center manifold. In the first case, the center manifold is $(n-1)$-dimensional and the analysis of the dynamics of the system can be reduced to that of analyzing the dynamics of a one-dimensional dynamical system. On the other hand, in the second case, the center manifold is $(n-2)$-dimensional and the analysis of the dynamics of the system can be reduced to that of analyzing a two-dimensional dynamical system. In Sections 4.5 and 4.6, we show how one can obtain such reduced dynamical systems. In Case 2, Hopf bifurcation can occur; and in Case 1, three types of static bifurcation can occur: saddle-node, transcritical, and pitchfork bifurcations. In Section 4.4.5, we consider Hopf bifurcation; and in Sections 4.4.1–4.4.4 we consider bifurcations of the one-dimensional dynamical system

$$\dot{x} = f(x;\mu) \tag{4.62}$$

where x and f are scalars and $f(x;\mu)$ is a smooth function of x and μ.
We assume that $(x = 0, \mu = 0)$ is a nonhyperbolic fixed point of (4.62); that is, $f(0;0) = 0$ and $f_x(0,0) = 0$. Expanding $f(x;\mu)$ in a Taylor series for small x and μ, we obtain

$$\begin{aligned}\dot{x} = {} & f_\mu \mu + \tfrac{1}{2}(f_{\mu\mu}\mu^2 + 2f_{x\mu}x\mu + f_{xx}x^2) \\ & + \tfrac{1}{6}(f_{\mu\mu\mu}\mu^3 + 3f_{x\mu\mu}x\mu^2 + 3f_{xx\mu}x^2\mu + f_{xxx}x^3) + \cdots \end{aligned} \tag{4.63}$$

Next, we consider the following four cases: (a) $f_\mu \neq 0$ and $f_{xx} \neq 0$; (b) $f_\mu \neq 0$, $f_{xx} = 0$, and $f_{xxx} \neq 0$; (c) $f_\mu = 0$ and $f_{xx} \neq 0$; and (d) $f_\mu = 0$ and $f_{xx} = 0$. We show below that the first case corresponds to saddle-node bifurcation, the second case corresponds to a nonbifurcation, the third case corresponds to transcritical bifurcation, and the fourth case corresponds to pitchfork bifurcation.

4.4.1
Saddle-Node Bifurcation

We consider the case $f_\mu \neq 0$ and $f_{xx} \neq 0$. As $x \to 0$ and $\mu \to 0$, (4.63) tends to

$$\dot{x} = f_\mu \mu + \tfrac{1}{2} f_{xx} x^2 + \cdots \tag{4.64}$$

When $f_\mu \mu \neq 0$, the fixed points of (4.64) are

$$x^* = \pm\sqrt{\frac{-2f_\mu \mu}{f_{xx}}} \tag{4.65}$$

which exist only when $f_\mu f_{xx}\mu < 0$. The eigenvalues associated with these two fixed points are

$$\lambda = \pm f_{xx}\sqrt{\frac{-2f_\mu \mu}{f_{xx}}} \tag{4.66}$$

It follows from (4.65) that there are two branches of fixed points in the neighborhood of $(x, \mu) = (0, 0)$ for $f_\mu \mu < 0$ if $f_{xx} > 0$ and for $f_\mu \mu > 0$ if $f_{xx} < 0$. Then, it follows from (4.66) that the upper branch is stable and the lower branch is unstable if $f_{xx} < 0$ and that the upper branch is unstable and the lower branch is stable if $f_{xx} > 0$. This bifurcation of the nonhyperbolic fixed point at the origin as μ passes through zero is called *fold* or *tangent* or *saddle-node bifurcation*.

Example 4.6

We let $f_\mu = 1$ and $f_{xx} = \alpha = \pm 1$ and consider the system

$$\dot{x} = f(x; \mu) = \mu + \alpha x^2 \tag{4.67}$$

When $\alpha = -1$, (4.67) does not have any fixed points for $\mu < 0$ but has the two nontrivial fixed points

$$x = \sqrt{\mu} \quad \text{and} \quad x = -\sqrt{\mu}$$

for $\mu > 0$. The Jacobian matrix has a single eigenvalue given by

$$\lambda = -2x$$

Thus, the fixed point $x = \sqrt{\mu}$ is a stable node because $\lambda < 0$, and the fixed point $x = -\sqrt{\mu}$ is an unstable node because $\lambda > 0$. In Figure 4.1a, we display the different fixed-point solutions of (4.67) and their stability in the $x - \mu$ space. We use broken and solid lines to depict branches of unstable and stable fixed points, respectively.

On the other hand, when $\alpha = 1$, (4.67) does not have any fixed points for $\mu > 0$ but has the two nontrivial fixed points

$$x = \sqrt{-\mu} \quad \text{and} \quad x = -\sqrt{-\mu}$$

for $\mu < 0$. The Jacobian matrix has a single eigenvalue given by

$$\lambda = 2x$$

Thus, the fixed point $x = \sqrt{-\mu}$ is an unstable node because $\lambda > 0$, and the fixed point $x = -\sqrt{-\mu}$ is a stable node because $\lambda < 0$. In Figure 4.1b, we display the different fixed-point solutions of (4.67) and their stability in the $x - \mu$ space.

We note that $f(x; \mu) = 0$ and $f_x(x; \mu) = 0$ at $(0, 0)$, and hence there is a nonhyperbolic fixed point at $\mu = 0$. Moreover, we note that there is a change in the number of fixed points from zero to two as μ passes through zero. Therefore, the origin of the $x - \mu$ space is a static bifurcation. It is clear from Figure 4.1 that the stable and unstable branches meet at the bifurcation point and have the same tangent. Therefore, this bifurcation is called a tangent bifurcation. Although branches of stable and unstable nodes meet at this bifurcation point, the tangent bifurcation is also called saddle-node bifurcation because, in higher-dimensional systems, branches of saddle points and stable nodes meet at such static bifurcation points.

Equation 4.67 is the *normal form for a generic saddle-node bifurcation of a fixed point of a continuous system*.

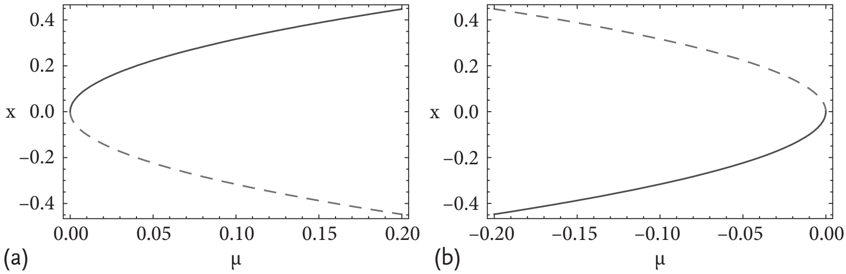

Figure 4.1 Scenario in the vicinity of a saddle-node bifurcation: (a) $\dot{x} = \mu - x^2$ and (b) $\dot{x} = \mu + x^2$.

4.4.2
Nonbifurcation Point

We consider the case $f_\mu \neq 0$, $f_{xx} = 0$, and $f_{xxx} \neq 0$. As $\mu \to 0$ and $x \to 0$, (4.63) tends to

$$\dot{x} = f_\mu \mu + \tfrac{1}{6} f_{xxx} x^3 + \cdots \tag{4.68}$$

It has only one nontrivial fixed point when $f_\mu \mu \neq 0$ given by

$$x^* = \left(\frac{-6 f_\mu \mu}{f_{xxx}} \right)^{1/3} \tag{4.69}$$

The eigenvalue associated with this fixed point is

$$\lambda = \frac{1}{2} f_{xxx} \left(\frac{-6 f_\mu \mu}{f_{xxx}} \right)^{2/3} \tag{4.70}$$

It follows from (4.69) that this fixed point exists on both sides of $f_\mu \mu$ and it follows from (4.70) that it is stable when $f_{xxx} < 0$ and unstable when $f_{xxx} > 0$. Although the fixed point (4.69) is nonhyperbolic because $f(x;\mu) = 0$ and $\lambda = 0$ at $\mu = 0$, this fixed point is not a bifurcation point because there is no qualitative change either in the number of fixed-point solutions or in the stability of the fixed-point solutions as μ passes through zero in the state-control space.

Example 4.7

We let $f_\mu = 1$ and $f_{xxx} = -6$ in (4.68) and consider

$$\dot{x} = f(x, \mu) = \mu - x^3 \tag{4.71}$$

We have only one fixed point, namely,

$$x = \mu^{1/3}$$

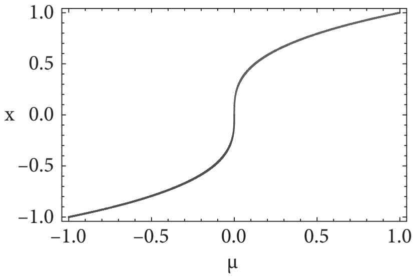

Figure 4.2 Fixed-point solutions of (4.71).

This solution is depicted in Figure 4.2. At the origin of the $x-\mu$ space, $f(x,\mu)=0$ and f_x has a zero eigenvalue, implying that $x=0$ is a nonhyperbolic fixed point at $\mu=0$. However, $(0,0)$ is not a bifurcation point because there is no qualitative change either in the number of fixed-point solutions or in the stability of the fixed-point solutions as μ passes through zero in the state-control space.

4.4.3 Transcritical Bifurcation

We consider the case $f_\mu=0$ and $f_{xx}\neq 0$. As $\mu\to 0$ and $x\to 0$, (4.63) tends to

$$\dot{x}=\tfrac{1}{2}(f_{\mu\mu}\mu^2+2f_{x\mu}x\mu+f_{xx}x^2)+\cdots \tag{4.72}$$

whose fixed points are

$$x_1^*=\frac{-f_{x\mu}+\sqrt{f_{x\mu}^2-f_{\mu\mu}f_{xx}}}{f_{xx}}\mu \quad\text{and}$$
$$x_2^*=\frac{-f_{x\mu}-\sqrt{f_{x\mu}^2-f_{\mu\mu}f_{xx}}}{f_{xx}}\mu \tag{4.73}$$

They exist when $f_{x\mu}^2-f_{\mu\mu}f_{xx}>0$. Their associated eigenvalues are

$$\lambda(x_1^*)=\sqrt{f_{x\mu}^2-f_{\mu\mu}f_{xx}}\,\mu \quad\text{and}\quad \lambda(x_2^*)=-\sqrt{f_{x\mu}^2-f_{\mu\mu}f_{xx}}\,\mu \tag{4.74}$$

Therefore, x_1^* is stable when $\mu<0$ and unstable when $\mu>0$. On the other hand, x_2^* is unstable when $\mu<0$ and stable when $\mu>0$. In this case, we have two branches of fixed points that interchange stability at $(0,0)$. Therefore, the bifurcation of the nonhyperbolic fixed point at the origin as μ passes through zero is transcritical bifurcation.

Example 4.8

We let $f_{\mu\mu} = 0$, $f_{\mu x} = 1$, and $f_{xx} = -2$ and consider the system

$$\dot{x} = f(x;\mu) = \mu x - x^2 \tag{4.75}$$

There are two fixed points:

$$x = 0\,; \quad \text{trivial fixed point}$$
$$x = \mu\,; \quad \text{nontrivial fixed point}$$

The Jacobian matrix

$$f_x = \mu - 2x$$

has the single eigenvalue

$$\lambda = \mu \quad \text{at } x = 0$$
$$\lambda = -\mu \quad \text{at } x = \mu$$

In the corresponding bifurcation diagram shown in Figure 4.3, the fixed point $x = 0$ is a nonhyperbolic fixed point at $\mu = 0$. At this point, a static bifurcation occurs because there is an exchange of stability between the trivial and nontrivial branches. The bifurcation point in Figure 4.3 is a *transcritical bifurcation point*. We point out that all of the branches that meet at this bifurcation point do not have the same tangent.

Equation 4.75 is the *normal form for a generic transcritical bifurcation of a fixed point of a continuous system.*

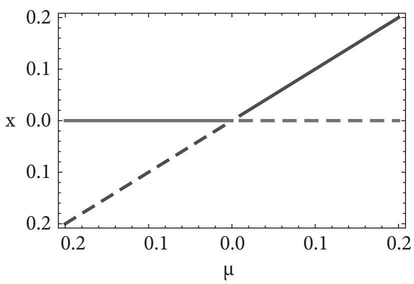

Figure 4.3 Scenario in the vicinity of a transcritical bifurcation.

4.4.4 Pitchfork Bifurcation

We consider the case $f_\mu = 0$ and $f_{xx} = 0$. As $\mu \to 0$ and $x \to 0$, (4.63) tends to

$$\dot{x} = f_{x\mu} x\mu + \tfrac{1}{6} f_{xxx} x^3 + \cdots \qquad (4.76)$$

whose fixed points are

$$x_1 = 0\,, \quad x_2 = \sqrt{\frac{-6 f_{x\mu}\mu}{f_{xxx}}}\,, \quad \text{and} \quad x_3 = -\sqrt{\frac{-6 f_{x\mu}\mu}{f_{xxx}}}$$

The second and third fixed points exist only when $f_{x\mu} f_{xxx}\mu < 0$. The Jacobian matrix in this case

$$f_x = f_{\mu x}\mu + \tfrac{1}{2} f_{xxx} x^2$$

has the single eigenvalue

$$\lambda(x_1) = f_{\mu x}\mu\,, \quad \lambda(x_2) = -2 f_{\mu x}\mu\,, \quad \lambda(x_3) = -2 f_{\mu x}\mu$$

Consequently, the trivial fixed point is stable when $f_{\mu x}\mu < 0$ and unstable when $f_{\mu x}\mu > 0$. On the other hand, the nontrivial fixed points exist and are stable when $f_{xxx} < 0$ and $f_{\mu x}\mu > 0$ and exist and are unstable when $f_{xxx} > 0$ and $f_{\mu x}\mu < 0$.

Example 4.9

We let $f_{\mu\mu} = 0$, $f_{\mu x} = 1$, and $f_{xxx}/6 = \alpha = \pm 1$ and consider the system

$$\dot{x} = f(x; \mu) = \mu x + \alpha x^3 \qquad (4.77)$$

where μ is again the scalar control parameter. There are three fixed points:

$$x = 0\,; \quad \text{trivial fixed point}$$
$$x = \pm\sqrt{-\mu/\alpha}\,; \quad \text{nontrivial fixed points}$$

In this case, the Jacobian matrix

$$f_x = \mu + 3\alpha x^2$$

has the single eigenvalue

$$\lambda = \mu \quad \text{at } x = 0$$
$$\lambda = -2\mu \quad \text{at } x = \pm\sqrt{-\mu/\alpha}$$

Consequently, the trivial fixed point is stable when $\mu < 0$ and unstable when $\mu > 0$. On the other hand, when $\alpha < 0$, nontrivial fixed points exist only when $\mu > 0$ and they are stable. However, when $\alpha > 0$, nontrivial fixed points exist only when

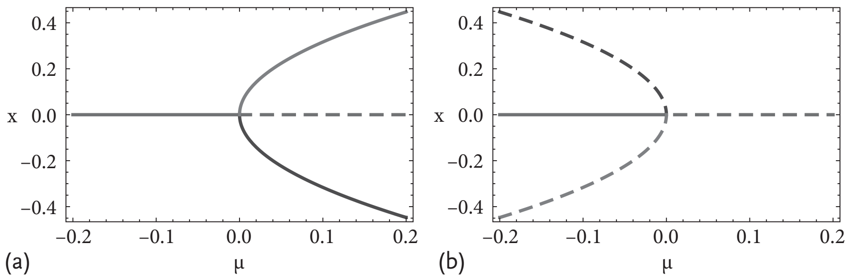

Figure 4.4 Scenario in the vicinity of a pitchfork bifurcation: (a) supercritical pitchfork bifurcation ($\alpha = -1$) and (b) subcritical pitchfork bifurcation ($\alpha = 1$).

$\mu < 0$ and they are unstable. The bifurcation diagrams of Figure 4.4a,b correspond to $\alpha = -1$ and $\alpha = 1$, respectively. In both cases, we note the following at $(0, 0)$: (a) $f(x, \mu) = 0$, (b) f_x has a zero eigenvalue, (c) the number of fixed-point solutions for $\mu < 0$ is different from that for $\mu > 0$, and (d) there is a change in the stability of the trivial fixed point as we pass through $\mu = 0$. Hence, the origin of the state-control space is a bifurcation point.

When $\alpha = -1$, two stable branches of fixed points $x = \sqrt{\mu}$ and $x = -\sqrt{\mu}$ bifurcate from the bifurcation point, as shown in Figure 4.4a. When $\alpha = 1$, two unstable branches of fixed points $x = \sqrt{\mu}$ and $x = -\sqrt{\mu}$ bifurcate from the bifurcation point, as shown in Figure 4.4b. The bifurcations observed in Figure 4.4a,b are called *pitchfork bifurcations* because the bifurcating nontrivial branches have the geometry of a pitchfork at $(0, 0)$. Specifically, the bifurcation in Figure 4.4a is called a *supercritical pitchfork bifurcation*, and the bifurcation in Figure 4.4b is called a *subcritical* or *reverse pitchfork bifurcation*. In the case of a supercritical pitchfork bifurcation, locally we have a branch of stable fixed points on one side of the bifurcation point and two branches of stable fixed points and a branch of unstable fixed points on the other side of the bifurcation point. In the case of a subcritical pitchfork bifurcation, locally we have two branches of unstable fixed points and a branch of stable fixed points on one side of the bifurcation point and a branch of unstable fixed points on the other side of the bifurcation point.

Equation 4.77 is the *normal form for a generic pitchfork bifurcation of a fixed point of a continuous system.*

4.4.5
Hopf Bifurcations

When a scalar control parameter μ is varied, a Hopf bifurcation of a fixed point of a system such as (4.17) is said to occur at $\mu = \mu_c$ if the following conditions (Marsden and McCracken, 1976) are satisfied:

1. $\boldsymbol{F}(\boldsymbol{x}^*;\mu_c) = 0$,
2. The matrix $D_{\boldsymbol{x}}\boldsymbol{F}$ has a pair of purely imaginary eigenvalues $\pm i\omega$ while all of its other eigenvalues have nonzero negative real parts at $(\boldsymbol{x}^*;\mu_c)$,
3. For $\mu \simeq \mu_c$, let the analytic continuation of the pair of purely imaginary eigenvalues be $\lambda_r \pm i\omega$. Then, $\text{Real}(d\lambda_r/d\mu) \neq 0$ at $\mu = \mu_c$. This condition implies a transversal or nonzero speed crossing of the imaginary axis and hence is called a *transversality condition*.

Again, the first two conditions imply that the fixed point undergoing the bifurcation is a nonhyperbolic fixed point. When all of the above three conditions are satisfied, a periodic solution of period $2\pi/\omega$ is born at $(\boldsymbol{x}^*;\mu_c)$; bifurcating periodic solutions can also occur when the transversality condition is not satisfied (e.g., Marsden and McCracken, 1976). It is to be noted that bifurcating periodic solutions can also occur under certain other degenerate conditions (e.g., Golubitsky and Schaeffer, 1985). In such cases, we have *degenerate Hopf bifurcations*.

The Hopf bifurcation is also called the *Poincaré–Andronov–Hopf bifurcation* (e.g., Wiggins, 1990) to give credit to the works of Poincaré and Andronov that preceded the work of Hopf. As pointed out in the literature (e.g., Abed, 1994; Arnold, 1988), Poincaré (1899) was aware of the conditions for this bifurcation to occur. (Poincaré studied such bifurcations in the context of lunar orbital dynamics.) Andronov and his coworkers studied Hopf bifurcations in planar systems before Hopf studied such bifurcations in general n-dimensional systems (Andronov *et al.*, 1966; Arnold, 1988). In aeroelasticity, the consequence of a Hopf bifurcation is known as *galloping* or *flutter*.

Example 4.10

We consider the planar system

$$\dot{x} = \mu x - \omega y + (\alpha x - \beta y)(x^2 + y^2) \tag{4.78}$$

$$\dot{y} = \omega x + \mu y + (\beta x + \alpha y)(x^2 + y^2) \tag{4.79}$$

where x and y are the states and μ is the scalar control parameter. The fixed point $(0,0)$ is a solution of (4.78) and (4.79) for all values of μ. The eigenvalues of the corresponding Jacobian matrix there are

$$\lambda_1 = \mu - i\omega \quad \text{and} \quad \lambda_2 = \mu + i\omega$$

From these eigenvalues, we note that $(0,0)$ is a nonhyperbolic fixed point of (4.78) and (4.79) when $\mu = 0$. Further, at $(x, y, \mu) = (0, 0, 0)$, we note that

$$\frac{d\lambda_1}{d\mu} = 1 \quad \text{and} \quad \frac{d\lambda_2}{d\mu} = 1$$

Hence, the three conditions required for a Hopf bifurcation are satisfied, and a Hopf bifurcation of the fixed point $(0, 0)$ of (4.78) and (4.79) occurs at $\mu = 0$. The period of the bifurcating periodic solution at $(0, 0, 0)$ is $2\pi/\omega$.

By using the transformation

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

we transform (4.78) and (4.79) into

$$\dot{r} = \mu r + \alpha r^3 \tag{4.80}$$

$$\dot{\theta} = \omega + \beta r^2 \tag{4.81}$$

The trivial fixed point of (4.80) corresponds to the fixed point $(0, 0)$ of (4.78) and (4.79), and a nontrivial fixed point (i.e., $r \neq 0$) of (4.80) corresponds to a periodic solution of (4.78) and (4.79). In the latter case, r is the amplitude and $\dot{\theta}$ is the frequency of the periodic solution that is created due to the Hopf bifurcation. A stable nontrivial fixed point of (4.80) corresponds to a stable periodic solution of (4.78) and (4.79). Likewise, an unstable nontrivial fixed point of (4.80) corresponds to an unstable periodic solution of (4.78) and (4.79).

We note that (4.80) is identical to (4.77), so the Hopf bifurcation at $(0, 0, 0)$ in the $x - y - \mu$ space is equivalent to a pitchfork bifurcation at $(0, 0)$ in the $r - \mu$ space. When $\alpha = -1$, we have a supercritical pitchfork bifurcation in the $r - \mu$ space and, hence, a *supercritical Hopf bifurcation* in the $x - y - \mu$ space. When $\alpha = 1$, we have a subcritical pitchfork bifurcation in the $r - \mu$ space and, hence, a *subcritical Hopf bifurcation* in the $x - y - \mu$ space. The bifurcation diagrams for $\alpha = -1$ are shown in Figure 4.5a,c and $\alpha = 1$ are shown in Figure 4.5b,d, respectively. In

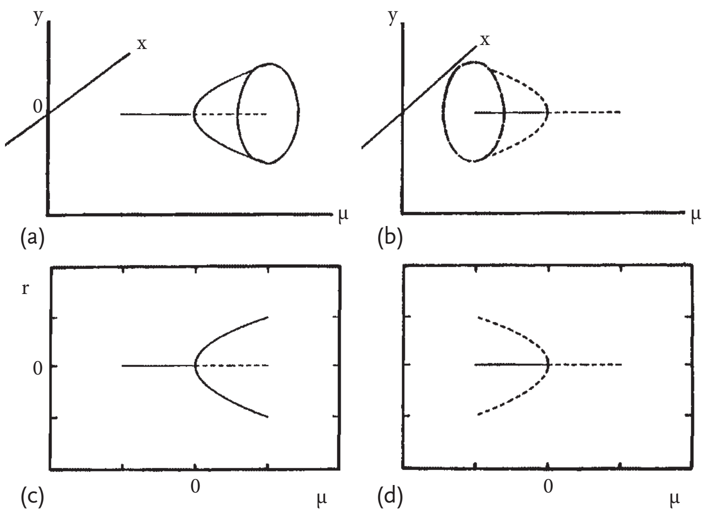

Figure 4.5 Local scenarios: (a,c) supercritical Hopf bifurcation and (b,d) subcritical Hopf bifurcation.

Figure 4.5a,b, the bifurcating periodic solutions in the $x - y - \mu$ space are depicted as parabolic surfaces. In the case of a supercritical Hopf bifurcation, locally we have a branch of stable fixed points on one side of the bifurcation point and a branch of unstable fixed points and a branch of stable periodic solutions on the other side of the bifurcation point. In the case of a subcritical Hopf bifurcation, locally we have a branch of unstable periodic solutions and a branch of stable fixed points on one side of the bifurcation point and a branch of unstable fixed points on the other side of the bifurcation point.

When $\alpha \neq 0$, (4.78) and (4.79) are the *normal form for a generic Hopf bifurcation of a fixed point of a continuous system*. On the other hand, when $\alpha = 0$ in (4.78) and (4.79), although the conditions for a Hopf bifurcation are satisfied, there are no periodic orbits in the vicinity of the bifurcation point to third order. This case is degenerate.

4.5 Normal Forms of Static Bifurcations

In this section, we consider reduction of the nonlinear continuous system (4.17) near a static bifurcation fixed point to its normal form. We assume that the fixed point has been shifted to $\boldsymbol{x} = 0$ and its corresponding control parameter has been shifted to $\mu = 0$. Moreover, we expand (4.17) in a Taylor series for small $\boldsymbol{x}$ and μ and obtain

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{b}\mu + B\mu\boldsymbol{x} + \boldsymbol{Q}(\boldsymbol{x}, \boldsymbol{x}) + \boldsymbol{C}(\boldsymbol{x}, \boldsymbol{x}, \boldsymbol{x}) + \cdots \tag{4.82}$$

where $A = D_{\boldsymbol{x}}\boldsymbol{F}$, $B = D_{\mu\boldsymbol{x}}\boldsymbol{F}$, and $\boldsymbol{b} = D_\mu\boldsymbol{F}$ at $(\boldsymbol{x}, \mu) = (\boldsymbol{0}, \mu)$ and $\boldsymbol{Q}(\boldsymbol{x}, \boldsymbol{x})$ and $\boldsymbol{C}(\boldsymbol{x}, \boldsymbol{x}, \boldsymbol{x})$ are bilinear and trilinear column vectors involving quadratic and cubic terms, respectively. Because the fixed point $(\boldsymbol{x}, \mu) = (\boldsymbol{0}, 0)$ is a static bifurcation point, one of the eigenvalues of the matrix A is zero and all of its other eigenvalues are in the left-half of the complex plane. Next, we demonstrate how one can use the method of multiple scales, a combination of center-manifold reduction and the method of normal forms, and a projection method to compute the normal forms of saddle-node, transcritical, and pitchfork bifurcations.

4.5.1 The Method of Multiple Scales

To compute the normal form of the static bifurcation of (4.82) at the origin, we introduce a small nondimensional parameter ϵ as a bookkeeping parameter and seek a third-order approximate solution of (4.82) in the form

$$\boldsymbol{x}(t; \mu) = \epsilon\boldsymbol{x}_1(T_0, T_1, T_2) + \epsilon^2\boldsymbol{x}_2(T_0, T_1, T_2) + \epsilon^3\boldsymbol{x}_3(T_0, T_1, T_2) + \cdots \tag{4.83}$$

where the time scales $T_m = \epsilon^m t$. The time derivative can be expressed in terms of these scales as

$$\frac{d}{dt} = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \cdots \tag{4.84}$$

where $D_m = \partial/\partial T_m$. Moreover, because μ is small, we expand it in terms of ϵ as

$$\mu = \epsilon\mu_1 + \epsilon^2\mu_2 + \epsilon^3\mu_3 + \cdots \tag{4.85}$$

Substituting (4.83)–(4.85) into (4.82) and equating coefficients of like powers of ϵ, we obtain

Order (ϵ)

$$D_0\boldsymbol{x}_1 - A\boldsymbol{x}_1 = \boldsymbol{b}\mu_1 \tag{4.86}$$

Order (ϵ^2)

$$D_0\boldsymbol{x}_2 - A\boldsymbol{x}_2 = \boldsymbol{b}\mu_2 - D_1\boldsymbol{x}_1 + B\boldsymbol{x}_1\mu_1 + \boldsymbol{Q}(\boldsymbol{x}_1, \boldsymbol{x}_1) \tag{4.87}$$

Order (ϵ^3)

$$\begin{aligned} D_0\boldsymbol{x}_3 - A\boldsymbol{x}_3 = {} & \boldsymbol{b}\mu_3 - D_1\boldsymbol{x}_2 - D_2\boldsymbol{x}_1 + B\boldsymbol{x}_2\mu_1 + B\boldsymbol{x}_1\mu_2 \\ & + 2\boldsymbol{Q}(\boldsymbol{x}_1, \boldsymbol{x}_2) + \boldsymbol{C}(\boldsymbol{x}_1, \boldsymbol{x}_1, \boldsymbol{x}_1) \end{aligned} \tag{4.88}$$

The general solution of (4.86) is the superposition of a homogeneous solution $\boldsymbol{x}_{1h}$ and a particular solution $\boldsymbol{x}_{1p}$. For the homogeneous solution, we denote the eigenvalues of the matrix A by λ_m, $m = 1, 2, \ldots, n$ and order them so that $\lambda_1 = 0$. We let $\boldsymbol{p}_m$ be the eigenvector (generalized eigenvector) corresponding to λ_m. Then, all of the terms corresponding to λ_m and $\boldsymbol{p}_m$ for $m > 2$ decay with time so that the long-time dynamics is given by

$$\boldsymbol{x}_{1h} = u(T_1, T_2)\boldsymbol{p} \tag{4.89}$$

where the subscript in $\boldsymbol{p}_1$ has been suppressed and the function $u(T_1, T_2)$ is determined by imposing solvability conditions at the higher-order approximations.

We note that A is singular because one of its eigenvalues is zero. Because the term $\boldsymbol{b}\mu_1$ on the right-hand side of (4.86) is independent of T_0, the particular solution of (4.86) is given by

$$A\boldsymbol{x}_{1p} = -\boldsymbol{b}\mu_1 \tag{4.90}$$

Because A is singular, (4.90) has a solution if and only if its right-hand side $-\boldsymbol{b}\mu_1$ is orthogonal to every solution of the adjoint homogeneous equation; that is solutions of

$$\boldsymbol{q}^{\mathrm{T}}A = 0$$

which has a nontrivial solution because A is singular. This condition demands that $\boldsymbol{q}^{\mathrm{T}}\boldsymbol{b}\mu_1 = 0$. There are two possibilities. First, $\boldsymbol{q}^{\mathrm{T}}\boldsymbol{b} \neq 0$ and hence $\mu_1 = 0$. Second, $\boldsymbol{q}^{\mathrm{T}}\boldsymbol{b} = 0$ and hence μ_1 is arbitrary. In the latter case, the solution is not unique. To make it unique, we require it to be orthogonal to the left eigenvector $\boldsymbol{q}$ and write the particular solution as

$$\boldsymbol{x}_{1p} = \boldsymbol{c}\mu_1 \quad \text{where} \quad A\boldsymbol{c} = -\boldsymbol{b} \tag{4.91}$$

and $\boldsymbol{q}^{\mathrm{T}}\boldsymbol{c} = 1$. We consider these two cases separately, starting with the first case.

The Case $\boldsymbol{q}^{\mathrm{T}}\boldsymbol{b} \neq \mathbf{0}$ In this case, the solution of the first-order problem, (4.86), can be expressed as

$$\boldsymbol{x}_1 = u(T_1, T_2)\boldsymbol{p} \tag{4.92}$$

Substituting (4.92) into (4.87) yields

$$D_0\boldsymbol{x}_2 - A\boldsymbol{x}_2 = \boldsymbol{b}\mu_2 - D_1 u\boldsymbol{p} + \boldsymbol{Q}(\boldsymbol{p}, \boldsymbol{p})u^2 \tag{4.93}$$

Again, the general solution of (4.93) is the superposition of a homogeneous term and a particular term. The homogeneous solution is the same as that given in (4.89) and hence is neglected. A particular solution of (4.93) is given by

$$A\boldsymbol{x}_2 = -\boldsymbol{b}\mu_2 + D_1 u\boldsymbol{p} - \boldsymbol{Q}(\boldsymbol{p}, \boldsymbol{p})u^2 \tag{4.94}$$

Equation 4.94 has a solution if and only if its right-hand side is orthogonal to $\boldsymbol{q}$; that is,

$$D_1 u = \boldsymbol{q}^{\mathrm{T}}\boldsymbol{b}\mu_2 + \boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{p}, \boldsymbol{p})u^2 \tag{4.95}$$

Hence, there is a generic saddle-node bifurcation at the origin of (4.82) if $\boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{p}, \boldsymbol{p}) \neq 0$.

When $\boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{p}, \boldsymbol{p}) = 0$, it follows from (4.95) that $\mu_2 = 0$ and $D_1 u = 0$ or $u = u(T_2)$. Then, a unique particular solution of (4.94) is given by

$$\boldsymbol{x}_2 = \boldsymbol{z}u^2 \quad \text{where} \quad A\boldsymbol{z} = -\boldsymbol{Q}(\boldsymbol{p}, \boldsymbol{p}) \quad \text{and} \quad \boldsymbol{q}^{\mathrm{T}}\boldsymbol{z} = 1 \tag{4.96}$$

Substituting (4.92) and (4.96) into (4.88) and using the fact that $\mu_1 = 0$, $\mu_2 = 0$, and $D_1 u = 0$, we obtain

$$D_0\boldsymbol{x}_3 - A\boldsymbol{x}_3 = \boldsymbol{b}\mu_3 - D_2 u\boldsymbol{p} + \left[2\boldsymbol{Q}(\boldsymbol{z}, \boldsymbol{p}) + \boldsymbol{C}(\boldsymbol{p}, \boldsymbol{p}, \boldsymbol{p})\right]u^3 \tag{4.97}$$

Equation 4.97 has a solution if and only if its right-hand side is orthogonal to $\boldsymbol{q}$; that is,

$$D_2 u = \boldsymbol{q}^{\mathrm{T}}\boldsymbol{b}\mu_3 + \left[2\boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{z}, \boldsymbol{p}) + \boldsymbol{q}^{\mathrm{T}}\boldsymbol{C}(\boldsymbol{p}, \boldsymbol{p}, \boldsymbol{p})\right]u^3 \tag{4.98}$$

Therefore, the origin of (4.82) is an inflection point and there is no bifurcation there if the coefficient of u^3 in (4.98) is different from zero.

The Case $\boldsymbol{q}^{\mathrm{T}}\boldsymbol{b} = \mathbf{0}$ In this case, the general solution of (4.86) can be expressed as

$$\boldsymbol{x}_1 = u(T_1, T_2)\boldsymbol{p} + \boldsymbol{c}\mu_1 \tag{4.99}$$

Substituting (4.99) into (4.87) yields

$$\begin{aligned} D_0\boldsymbol{x}_2 - A\boldsymbol{x}_2 = \boldsymbol{b}\mu_2 - \boldsymbol{p}D_1 u + \left[B\boldsymbol{p} + 2\boldsymbol{Q}(\boldsymbol{c}, \boldsymbol{p})\right]\mu_1 u \\ + \left[B\boldsymbol{c} + \boldsymbol{Q}(\boldsymbol{c}, \boldsymbol{c})\right]\mu_1^2 + \boldsymbol{Q}(\boldsymbol{p}, \boldsymbol{p})u^2 \end{aligned} \tag{4.100}$$

The solvability condition of (4.100) demands that its right-hand side be orthogonal to $\boldsymbol{q}$, which yields

$$D_1 u = \left[\boldsymbol{q}^{\mathrm{T}} B \boldsymbol{c} + \boldsymbol{q}^{\mathrm{T}} \boldsymbol{Q}(\boldsymbol{c}, \boldsymbol{c})\right] \mu_1^2 + \left[\boldsymbol{q}^{\mathrm{T}} B \boldsymbol{p} + 2\boldsymbol{q}^{\mathrm{T}} \boldsymbol{Q}(\boldsymbol{c}, \boldsymbol{p})\right] \mu_1 u + \boldsymbol{q}^{\mathrm{T}} \boldsymbol{Q}(\boldsymbol{p}, \boldsymbol{p}) u^2 \tag{4.101}$$

Therefore, there is a generic transcritical bifurcation of (4.82) at the origin when $\boldsymbol{q}^{\mathrm{T}} \boldsymbol{Q}(\boldsymbol{p}, \boldsymbol{p}) \neq 0$.

When $\boldsymbol{q}^{\mathrm{T}} \boldsymbol{Q}(\boldsymbol{p}, \boldsymbol{p}) = 0$, it follows from (4.101) that $\mu_1 = 0$ and $D_1 u = 0$ and hence $u = u(T_2)$. Then, the solution of (4.100) can be expressed as

$$\boldsymbol{x}_2 = \boldsymbol{c}\mu_2 + \boldsymbol{z} u^2 \tag{4.102}$$

where $\boldsymbol{c}$ and $\boldsymbol{z}$ are defined in (4.91) and (4.96), respectively. Then, substituting (4.99) and (4.102) into (4.88) and using the fact that $\mu_1 = 0$ and $D_1 u = 0$, we obtain

$$\begin{aligned} D_0 \boldsymbol{x}_3 - A \boldsymbol{x}_3 = \boldsymbol{b}\mu_3 - D_2 u \boldsymbol{p} + \left[B \boldsymbol{p} + 2\boldsymbol{Q}(c, \boldsymbol{p})\right] \mu_2 u \\ + \left[2\boldsymbol{Q}(\boldsymbol{z}, \boldsymbol{p}) + \boldsymbol{C}(\boldsymbol{p}, \boldsymbol{p}, \boldsymbol{p})\right] u^3 \end{aligned} \tag{4.103}$$

Imposing the solvability condition in (4.103) yields

$$D_2 u = \left[\boldsymbol{q}^{\mathrm{T}} B \boldsymbol{p} + 2\boldsymbol{q}^{\mathrm{T}} \boldsymbol{Q}(c, \boldsymbol{p})\right] \mu_2 u + \left[2\boldsymbol{q}^{\mathrm{T}} \boldsymbol{Q}(\boldsymbol{z}, \boldsymbol{p}) + \boldsymbol{q}^{\mathrm{T}} \boldsymbol{C}(\boldsymbol{p}, \boldsymbol{p}, \boldsymbol{p})\right] u^3 \tag{4.104}$$

Therefore, there is a generic supercritical pitchfork bifurcation at the origin of (4.82) when $2\boldsymbol{q}^{\mathrm{T}} \boldsymbol{Q}(\boldsymbol{z}, \boldsymbol{p}) + \boldsymbol{q}^{\mathrm{T}} \boldsymbol{C}(\boldsymbol{p}, \boldsymbol{p}, \boldsymbol{p}) < \boldsymbol{0}$ and a generic subcritical pitchfork bifurcation at the origin of (4.82) when $2\boldsymbol{q}^{\mathrm{T}} \boldsymbol{Q}(\boldsymbol{z}, \boldsymbol{p}) + \boldsymbol{q}^{\mathrm{T}} \boldsymbol{C}(\boldsymbol{p}, \boldsymbol{p}, \boldsymbol{p}) > \boldsymbol{0}$.

Example 4.11

We construct the normal form of the system

$$\dot{x}_1 = b_1 \mu + 2x_1 - 2x_2 + \mu x_1 + a_{11} x_1^2 + a_{12} x_1 x_2 + a_{13} x_2^2 \tag{4.105}$$

$$\dot{x}_2 = b_2 \mu + 4x_1 - 4x_2 - 4\mu x_2 + a_{21} x_1^2 + a_{22} x_1 x_2 + a_{23} x_2^2 \tag{4.106}$$

near the fixed point $(x_1, x_2, \mu) = (0, 0, 0)$. We seek a third-order approximate solution of (4.105) and (4.106) in the form

$$x_n(t; \mu) = \epsilon x_{n1}(T_0, T_1, T_2) + \epsilon^2 x_{n2}(T_0, T_1, T_2) + \epsilon^3 x_{n3}(T_0, T_1, T_2) + \cdots \tag{4.107}$$

Substituting (4.107), (4.84), and (4.85) into (4.105) and (4.106) and equating coefficients of equal powers of ϵ, we obtain

Order (ϵ)

$$D_0x_{11} - 2x_{11} + 2x_{21} = \mu_1 b_1 \tag{4.108}$$

$$D_0x_{21} - 4x_{11} + 4x_{21} = \mu_1 b_2 \tag{4.109}$$

Order (ϵ^2)

$$D_0x_{12} - 2x_{12} + 2x_{22} = \mu_2 b_1 - D_1x_{11} + \mu_1 x_{11} + a_{11}x_{11}^2 + a_{12}x_{11}x_{21} + a_{13}x_{21}^2 \tag{4.110}$$

$$D_0x_{22} - 4x_{12} + 4x_{22} = \mu_2 b_2 - D_1x_{21} - 4\mu_1 x_{21} + a_{21}x_{11}^2 + a_{22}x_{11}x_{21} + a_{23}x_{21}^2 \tag{4.111}$$

Order (ϵ^3)

$$D_0x_{13} - 2x_{13} + 2x_{23} = \mu_3 b_1 - D_1x_{12} - D_2x_{11} + \mu_1 x_{12} + \mu_2 x_{11} + 2a_{11}x_{11}x_{12} + a_{12}x_{11}x_{22} + a_{12}x_{12}x_{21} + 2a_{13}x_{21}x_{22} \tag{4.112}$$

$$D_0x_{23} - 4x_{13} + 4x_{23} = \mu_3 b_2 - D_1x_{23} - D_2x_{21} - 4\mu_1 x_{23} - 4\mu_2 x_{21} + 2a_{21}x_{11}x_{13} + a_{23}x_{11}x_{23} + a_{23}x_{13}x_{21} + 2a_{23}x_{21}x_{23} \tag{4.113}$$

In this case, the coefficient matrix of (4.108) and (4.109) is

$$A = \begin{bmatrix} 2 & -2 \\ 4 & -4 \end{bmatrix} \tag{4.114}$$

whose eigenvalues are $\lambda = 0$ and $\lambda = -2$. Hence, the origin of (4.105) and (4.106) may be a static bifurcation point. The right and left eigenvectors of A corresponding to $\lambda = 0$ are

$$\boldsymbol{p} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{q} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

where $\boldsymbol{q}^{\mathrm{T}}\boldsymbol{p} = 1$. Because A is singular, (4.108) and (4.109) have a solution if and only if their right-hand sides are orthogonal to $\boldsymbol{q}^{\mathrm{T}}$; that is, if and only if $(2b_1 - b_2)\mu_1 = 0$. There are two possibilities: (a) $(2b_1 - b_2) \neq 0$ and hence $\mu_1 = 0$ and (a) $(2b_1 - b_2) = 0$ and hence μ_1 is arbitrary. We treat both cases separately, starting with the first case.

When $(2b_1 - b_2) \neq 0$, the solution of (4.108) and (4.109) can be expressed as

$$x_{11} = u(T_1, T_2) \quad \text{and} \quad x_{21} = u(T_1, T_2) \tag{4.115}$$

Substituting (4.115) into (4.110) and (4.111) yields

$$D_0 x_{12} - 2x_{12} + 2x_{22} = \mu_2 b_1 - D_1 u + (a_{11} + a_{12} + a_{13})u^2 \tag{4.116}$$

$$D_0 x_{22} - 4x_{12} + 4x_{22} = \mu_2 b_2 - D_1 u + (a_{21} + a_{22} + a_{23})u^2 \tag{4.117}$$

Demanding that the right-hand side of (4.116) and (4.117) be orthogonal to $\boldsymbol{q}$ yields the normal form

$$D_1 u = (2b_1 - b_2)\mu_2 + (2a_{11} + 2a_{12} + 2a_{13} - a_{21} - a_{22} - a_{23})u^2 \tag{4.118}$$

which indicates that the origin of (4.105) and (4.106) undergoes a generic saddle-node bifurcation as μ passes through zero if the coefficient of u^2 is different from zero. When this coefficient is zero, one needs to carry out the expansion to higher order.

When $(2b_1 - b_2) = 0$, the solution of (4.108) and (4.109) that is orthogonal to $\boldsymbol{q}$ can be expressed as

$$x_{11} = u(T_1, T_2) + \tfrac{1}{2} b_1 \mu_1 \quad \text{and} \quad x_{21} = u(T_1, T_2) + b_1 \mu_1 \tag{4.119}$$

Substituting (4.119) into (4.110) and (4.111) yields

$$\begin{aligned} &D_0 x_{12} - 2x_{12} + 2x_{22} \\ &= \mu_2 b_1 - D_1 u + \left(\tfrac{1}{2} + \tfrac{1}{4} a_{11} b_1 + \tfrac{1}{2} a_{12} b_1 + a_{13} b_1\right) b_1 \mu_1^2 \\ &\quad + \left(1 + a_{11} b_1 + \tfrac{3}{2} a_{12} b_1 + 2a_{13} b_1\right) \mu_1 u + (a_{11} + a_{12} + a_{13})\, u^2 \end{aligned} \tag{4.120}$$

$$\begin{aligned} &D_0 x_{22} - 4x_{12} + 4x_{22} \\ &= \mu_2 b_2 - D_1 u + \left(-4 + \tfrac{1}{4} a_{21} b_1 + \tfrac{1}{2} a_{22} b_1 + a_{23} b_1\right) b_1 \mu_1^2 \\ &\quad + \left(-4 + a_{21} b_1 + \tfrac{3}{2} a_{22} b_1 + 2a_{23} b_1\right) \mu_1 u + (a_{21} + a_{22} + a_{23}) u^2 \end{aligned} \tag{4.121}$$

The solvability condition of (4.120) and (4.121) yields the following normal form of (4.105) and (4.106):

$$D_1 u = \Gamma_0 \mu_1^2 + \Gamma_1 \mu_1 u + \Gamma_2 u^2 \tag{4.122}$$

where

$$\Gamma_0 = \tfrac{1}{4} b_1 (20 + 2a_{11} b_1 + 4a_{12} b_1 + 8a_{13} b_1 - a_{21} b_1 - 2a_{22} b_1 - 4a_{23} b_1) \tag{4.123}$$

$$\Gamma_1 = \tfrac{1}{2} (12 + 4a_{11} b_1 + 6a_{12} b_1 + 8a_{13} b_1 - 2a_{21} b_1 - 3a_{22} b_1 - 4a_{23} b_1) \tag{4.124}$$

$$\Gamma_2 = 2a_{11} + 2a_{12} + 2a_{13} - a_{21} - a_{22} - a_{23} \tag{4.125}$$

Equation 4.122 indicates that the origin of (4.105) and (4.106) undergoes a generic transcritical bifurcation as μ passes through zero if $\Gamma_2 \neq 0$.

When $\Gamma_2 = 0$, it follows from (4.122) that $\mu_1 = 0$ and $D_1 u = 0$ or $u = u(T_2)$. Then, the particular solution of (4.120) and (4.121) that is orthogonal to $\boldsymbol{q}^{\mathrm{T}}$ can be expressed as

$$x_{12} = \tfrac{1}{2} b_1 \mu_2 + \tfrac{1}{2}(a_{11} + a_{12} + a_{13}) u^2 \quad \text{and} \quad x_{22} = b_1 \mu_2 + (a_{11} + a_{12} + a_{13}) u^2 \tag{4.126}$$

Substituting (4.119) and (4.126) into (4.112) and (4.113) and recalling that $\mu_1 = 0$ and $D_1 u = 0$, we obtain

$$\begin{aligned} &D_0 x_{13} - 2x_{13} + 2x_{23} \\ &= \mu_3 b_1 - D_2 u + \tfrac{1}{2}(2 + 2a_{11} b_1 + 3a_{12} b_1 + 4a_{13} b_1)\mu_2 u \\ &\quad + \tfrac{1}{4}(2a_{11} + 3a_{12} + 4a_{13})(a_{21} + a_{22} + a_{23}) u^3 \end{aligned} \tag{4.127}$$

$$\begin{aligned} &D_0 x_{23} - 4x_{13} + 4x_{23} \\ &= \mu_3 b_2 - D_2 u - \tfrac{1}{2}(8 - 2a_{21} b_1 - 3a_{23} b_1 - 4a_{23} b_1)\mu_2 u \\ &\quad + \tfrac{1}{4}(a_{21} + a_{23} + a_{23})(2a_{21} + 3a_{23} + 4a_{23}) u^3 \end{aligned} \tag{4.128}$$

Imposing the solvability condition for (4.127) and (4.128) yields

$$D_2 u = \nu \mu_2 u + \alpha u^3 \tag{4.129}$$

where

$$\begin{aligned} \nu &= \tfrac{1}{2}(12 + 4a_{11} b_1 + 6a_{12} b_1 + 8a_{13} b_1 - 2a_{21} b_1 - 3a_{22} b_1 - 4a_{23} b_1) \\ \alpha &= \tfrac{1}{4}(4a_{11} + 6a_{12} + 8a_{13} - 2a_{21} - 3a_{22} - 4a_{23})(a_{21} + a_{22} + a_{23}) \end{aligned}$$

It follows from (4.139) that, as μ passes through zero, the origin of (4.105) and (4.106) undergoes a generic supercritical pitchfork bifurcation when $\alpha < 0$ and a generic subcritical pitchfork bifurcation when $\alpha > 0$.

Example 4.12

We consider the three-dimensional dynamic system

$$\dot{x}_1 = b_1 \mu - x_1 + x_2 - 2x_3 + \mu x_1 + \alpha_1 (x_1 + 2x_2 + x_3)^2 \tag{4.130}$$

$$\dot{x}_2 = b_2 \mu - 2x_1 - x_2 - x_3 + \mu x_2 + \alpha_2 (x_1 - x_2 + x_3)^2 \tag{4.131}$$

$$\dot{x}_3 = b_3 \mu + x_1 + 2x_2 - x_3 - 2\mu x_3 + \alpha_3 (2x_1 + x_2 - x_3)^2 \tag{4.132}$$

Substituting (4.107), (4.84), and (4.85) into (4.130)–(4.132) and equating coefficients of like powers of ϵ yields.

Order (ϵ)

$$D_0 x_{11} + x_{11} - x_{21} + 2x_{31} = b_1 \mu_1 \tag{4.133}$$

$$D_0 x_{21} + 2x_{11} + x_{21} + x_{31} = b_2 \mu_1 \tag{4.134}$$

$$D_0 x_{31} - x_{11} - 2x_{21} + x_{31} = b_3 \mu_1 \tag{4.135}$$

Order (ϵ^2)

$$\begin{aligned} D_0 x_{12} + x_{12} - x_{22} + 2x_{32} = &-D_1 x_{11} + b_1 \mu_2 + \mu_1 x_{11} \\ &+ \alpha_1 (x_{11} + 2x_{21} + x_{31})^2 \end{aligned} \tag{4.136}$$

$$\begin{aligned} D_0 x_{22} + 2x_{12} + x_{22} + x_{32} = &-D_1 x_{21} + b_2 \mu_2 + \mu_1 x_{21} \\ &+ \alpha_2 (x_{11} - x_{21} + x_{31})^2 \end{aligned} \tag{4.137}$$

$$\begin{aligned} D_0 x_{32} - x_{12} - 2x_{22} + x_{32} = &-D_1 x_{31} + b_3 \mu_2 - 2\mu_1 x_{31} \\ &+ \alpha_3 (2x_{11} + x_{21} - x_{31})^2 \end{aligned} \tag{4.138}$$

Order (ϵ^3)

$$\begin{aligned} D_0 x_{13} + x_{13} - x_{23} + 2x_{33} = &-D_1 x_{12} - D_2 x_{11} + b_1 \mu_3 + \mu_2 x_{11} + \mu_1 x_{12} \\ &+ 2\alpha_1 (x_{11} + 2x_{21} + x_{31}) (x_{12} + 2x_{22} + x_{32}) \end{aligned} \tag{4.139}$$

$$\begin{aligned} D_0 x_{23} + 2x_{13} + x_{23} + x_{33} = &-D_1 x_{22} - D_2 x_{21} + b_2 \mu_3 + \mu_2 x_{21} + \mu_1 x_{22} \\ &+ 2\alpha_2 (x_{11} - x_{21} + x_{31}) (x_{12} - x_{22} + x_{32}) \end{aligned} \tag{4.140}$$

$$\begin{aligned} D_0 x_{33} - x_{13} - 2x_{23} + x_{33} = &-D_1 x_{32} - D_2 x_{31} + b_3 \mu_3 - 2\mu_2 x_{31} - 2\mu_1 x_{32} \\ &+ 2\alpha_3 (2x_{11} + x_{21} - x_{31}) (2x_{12} + x_{22} - x_{32}) \end{aligned} \tag{4.141}$$

In this case, the coefficient matrix of (4.133)–(4.135) is

$$\begin{pmatrix} -1 & 1 & -2 \\ -2 & -1 & -1 \\ 1 & 2 & -1 \end{pmatrix} \tag{4.142}$$

whose eigenvalues are $\lambda = 0$ and $\lambda = 3/2(-1 \pm i\sqrt{3})$. Hence, the origin of (4.130)–(4.133) may be a static bifurcation point. The right and left eigenvectors of A corresponding to $\lambda = 0$ are

$$\boldsymbol{p} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{q} = \frac{1}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

where $\boldsymbol{q}^{\mathrm{T}}\boldsymbol{p} = 1$. Because A is singular, (4.133)–(4.135) have a solution if and only if their right-hand sides are orthogonal to $\boldsymbol{q}^{\mathrm{T}}$; that is, if and only if $(b_3+b_2-b_1)\mu_1 = 0$. There are two possibilities: (a) $(b_3 + b_2 - b_1) \neq 0$ and hence $\mu_1 = 0$ and (a) $(b_3 + b_2 - b_1) = 0$ and hence μ_1 is arbitrary. We treat both cases separately, starting with the first case.

When $(b_3 + b_2 - b_1) \neq 0$, the nondecaying solution of (4.133)–(4.135) can be expressed as

$$x_{11} = -u(T_1, T_2)\,, \quad x_{21} = u(T_1, T_2)\,, \quad \text{and} \quad x_{31} = u(T_1, T_2) \tag{4.143}$$

Substituting (4.143) into (4.136)–(4.138) yields

$$D_0 x_{12} + x_{12} - x_{22} + 2x_{32} = D_1 u + 4u^2\alpha_1 + b_1\mu_2 \tag{4.144}$$

$$D_0 x_{22} + 2x_{12} + x_{22} + x_{32} = -D_1 u + u^2\alpha_2 + b_2\mu_2 \tag{4.145}$$

$$D_0 x_{32} - x_{12} - 2x_{22} + x_{32} = -D_1 u + 4u^2\alpha_3 + b_3\mu_2 \tag{4.146}$$

Demanding that the right-hand side of (4.144)–(4.146) be orthogonal to $\boldsymbol{q}$ yields the normal form

$$D_1 u = \tfrac{1}{3}\left(b_3 + b_2 - b_1\right)\mu_2 + \tfrac{1}{3}\left(4\alpha_3 + \alpha_2 - 4\alpha_1\right)u^2 \tag{4.147}$$

which indicates that the origin of (4.130)–(4.132) undergoes a generic saddle-node bifurcation as μ passes through zero if the coefficient of u^2 is different from zero. When this coefficient is zero, one needs to carry out the expansion to higher order.

When $(b_3 + b_2 - b_1) = 0$, the solution of (4.133)–(4.135) that is orthogonal to $\boldsymbol{q}$ can be expressed as

$$\begin{aligned} x_{11} &= -u(T_1, T_2) + \tfrac{1}{3}b_2\mu_1 \\ x_{21} &= u(T_1, T_2) - \tfrac{1}{3}b_3\mu_1 \\ x_{31} &= u(T_1, T_2) + \tfrac{1}{3}(b_3 + b_2)\mu_1 \end{aligned} \tag{4.148}$$

To simplify the algebra, we let $b_1 = 2b$, $b_2 = b$, and $b_3 = b$. Substituting (4.148) into (4.136)–(4.138) yields

$$\begin{aligned} D_0 x_{12} + x_{12} - x_{22} + 2x_{32} = {} & D_1 u + \tfrac{1}{9}b\left(3 + b\alpha_1\right)\mu_1^2 + 2b\mu_2 \\ & - \tfrac{1}{3}\left(3 - 4b\alpha_1\right)\mu_1 u + 4\alpha_1 u^2 \end{aligned} \tag{4.149}$$

$$\begin{aligned} D_0 x_{22} + 2x_{12} + x_{22} + x_{32} = {} & -D_1 u - \tfrac{1}{9}b\left(3 - 16b\alpha_2\right)\mu_1^2 + b\mu_2 \\ & + \tfrac{1}{3}\left(3 - 8b\alpha_2\right)\mu_1 u + \alpha_2 u^2 \end{aligned} \tag{4.150}$$

$$\begin{aligned} D_0 x_{32} - x_{12} - 2x_{22} + x_{32} = {} & -D_1 u - \tfrac{1}{9}b\left(12 - b\alpha_3\right)\mu_1^2 + b\mu_2 \\ & - \tfrac{2}{3}\left(3 - 2b\alpha_3\right)\mu_1 u + 4\alpha_3 u^2 \end{aligned} \tag{4.151}$$

The solvability condition of (4.149)–(4.151) yields the normal form of (4.130)–(4.132) as

$$D_1 u = \Gamma_0\mu_1^2 + \Gamma_1\mu_1 u + \Gamma_2 u^2 \tag{4.152}$$

where

$$\Gamma_0 = -\tfrac{1}{27}b\,(18 + b\alpha_1 - 16b\alpha_2 - b\alpha_3)$$
$$\Gamma_1 = \tfrac{4}{9}b\,(\alpha_3 - 2\alpha_2 - \alpha_1)$$
$$\Gamma_2 = \tfrac{1}{3}\,(4\alpha_3 + \alpha_2 - 4\alpha_1)$$

Equation 4.152 indicates that the origin of (4.130)–(4.132) undergoes a generic transcritical bifurcation as μ passes through zero if the coefficient of u^2 is different from zero; that is, $(4\alpha_3 + \alpha_2 - 4\alpha_1) \neq 0$.

When $\Gamma_2 = 0$, it follows from (4.152) that $\mu_1 = 0$ and $D_1 u = 0$ or $u = u(T_2)$. Then, the particular solution of (4.144)–(4.146) that is orthogonal to $\boldsymbol{q}^{\mathrm{T}}$ can be expressed as

$$\begin{aligned} x_{12} &= \tfrac{1}{3}b\mu_2 + \tfrac{1}{3}\alpha_2 u^2 \\ x_{22} &= -\tfrac{1}{3}b\mu_2 - \tfrac{4}{3}\alpha_3 u^2 \\ x_{32} &= \tfrac{2}{3}b\mu_2 + \left(\tfrac{1}{3}\alpha_2 + \tfrac{4}{3}\alpha_3\right) u^2 \end{aligned} \tag{4.153}$$

Substituting (4.148) and (4.153) into (4.139)–(4.141) and recalling that $\mu_1 = 0$ and $D_1 u = 0$, we obtain

$$\begin{aligned} D_0 x_{13} + x_{13} - x_{23} + 2x_{33} = {} & D_2 u + 2b\mu_3 + \tfrac{1}{3}\,(4b\alpha_3 + b\alpha_2 - 3)\,\mu_2 u \\ & + \tfrac{2}{3}(\alpha_2 - 2\alpha_3)(\alpha_2 + 4\alpha_3)u^3 \end{aligned} \tag{4.154}$$

$$\begin{aligned} D_0 x_{23} + 2x_{13} + x_{23} + x_{33} = {} & -D_2 u + b\mu_3 - \tfrac{1}{3}\,(8b\alpha_2 - 3)\,\mu_2 u \\ & - \tfrac{4}{3}\alpha_2\,(\alpha_2 + 4\alpha_3)\,u^3 \end{aligned} \tag{4.155}$$

$$\begin{aligned} D_0 x_{32} - x_{12} - 2x_{22} + x_{32} = {} & -D_2 u + b\mu_3 + \tfrac{2}{3}\,(2b\alpha_3 - 3)\,\mu_2 u \\ & + \tfrac{4}{3}\,(8\alpha_3 - \alpha_2)\,\alpha_3 u^3 \end{aligned} \tag{4.156}$$

Demanding that the right-hand side of (4.154)–(4.156) be orthogonal to $\boldsymbol{q}$ and using the fact that $\alpha_1 = 1/4\alpha_2 + \alpha_3$ yields the normal form

$$D_2 u = -b\alpha_2\mu_2 u + \tfrac{1}{3}\left(16\alpha_3^2 - 8\alpha_2\alpha_3 - 2\alpha_2^2\right) u^3 \tag{4.157}$$

It follows from (4.157) that the origin of (4.130)–(4.132) undergoes a supercritical pitchfork bifurcation as μ passes through zero when

$$\left(16\alpha_3^2 - 8\alpha_2\alpha_3 - 2\alpha_2^2\right) < 0$$

and a subcritical pitchfork bifurcation when $(16\alpha_3^2 - 8\alpha_2\alpha_3 - 2\alpha_2^2) > 0$.

4.5.2
Center-Manifold Reduction

To compute the normal form of the static bifurcation of (4.82) at the origin by using center-manifold reduction, we first calculate the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$

and eigenvectors (generalized eigenvectors) $\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n$ of the matrix A. Second, we arrange the eigenvalues of A so that $\lambda_1 = 0$ and let $\boldsymbol{p}_1$ be be its corresponding eigenvector. Third, we introduce the transformation $\boldsymbol{x} = P\boldsymbol{y}$ and obtain

$$\dot{\boldsymbol{y}} = J\boldsymbol{y} + P^{-1}\boldsymbol{b}\mu + P^{-1}BP\boldsymbol{y}\mu + P^{-1}\boldsymbol{Q}(P\boldsymbol{y}, P\boldsymbol{y}) + P^{-1}\boldsymbol{C}(P\boldsymbol{y}, P\boldsymbol{y}, P\boldsymbol{y}) \tag{4.158}$$

where $J = P^{-1}AP$. We note that J can be rewritten as

$$J = \begin{bmatrix} 0 & 0 \\ 0 & J_s \end{bmatrix} \tag{4.159}$$

where J_s is an $(n-1)\times(n-1)$ matrix whose eigenvalues are $\lambda_2, \lambda_3, \ldots, \lambda_n$. Fourth, we let $\boldsymbol{y}_s$ be the $(n-1)$-dimensional vector with the components $y_2, y_3, \ldots, y_n$ and rewrite (4.158) as

$$\dot{y}_1 = \hat{b}_1\mu + \sum_{i=1}^{n} \xi_i y_i \mu + f_2(y_1, \boldsymbol{y}_s) + f_3(y_1, \boldsymbol{y}_s) \tag{4.160}$$

and

$$\dot{\boldsymbol{y}}_s = J_s\boldsymbol{y}_s + \hat{\boldsymbol{b}}_s\mu + \boldsymbol{\zeta} y_1\mu + \hat{B}\boldsymbol{y}\mu + \boldsymbol{F}_2(y_1, \boldsymbol{y}_s) + \boldsymbol{F}_3(y_1, \boldsymbol{y}_s) \tag{4.161}$$

where $\hat{\boldsymbol{b}} = P^{-1}\boldsymbol{b}$, f_2 and $\boldsymbol{F}_2$ are the first and last $(n-1)$ components of $P^{-1}\boldsymbol{Q}(P\boldsymbol{y}, P\boldsymbol{y})$, and f_3 and $\boldsymbol{F}_3$ are the first and last $(n-1)$ components of $P^{-1}\boldsymbol{C}(P\boldsymbol{y}, P\boldsymbol{y}, P\boldsymbol{y})$.

To determine the dependence of the center manifold on μ, we augment (4.160) and (4.161) with the additional equation

$$\dot{\mu} = 0 \tag{4.162}$$

Because the f_i and $\boldsymbol{F}_i$ are polynomials and hence infinitely differentiable, there exists a *local center manifold* of the form (Carr, 1981)

$$\boldsymbol{y}_s = \boldsymbol{h}(y_1; \mu) \tag{4.163}$$

where $\boldsymbol{h}$ is a polynomial function of y_1 and μ such that

$$\boldsymbol{h}(0,0) = \boldsymbol{0}, \quad D_{y_1}h_i(0;0) = 0\,, \quad \text{and} \quad D_\mu h_i(0;0) = 0 \tag{4.164}$$

where the h_i are the scalar components of $\boldsymbol{h}$.

Substituting (4.163) into (4.161) yields

$$\begin{aligned} &\boldsymbol{h}'(y_1)\left[\hat{b}_1\mu + \xi_1 y_1\mu + \sum_{i=2}^{n} \xi_i h_i(y_1)\mu + f_2(y_1, \boldsymbol{h}(y_1))\right] \\ &\quad = J_s\boldsymbol{h}(y_1) + \hat{\boldsymbol{b}}_s\mu + \zeta y_1\mu + \hat{B}\boldsymbol{h}(y_1)\mu + \boldsymbol{F}_2[y_1, \boldsymbol{h}(y_1)] + \cdots \end{aligned} \tag{4.165}$$

To solve (4.165), one approximates the components of $\boldsymbol{h}(y_1;\mu)$ with polynomials. The polynomial approximations are usually taken to be quadratic to the first approximation and do not contain constant and linear terms so that the conditions (4.164) are satisfied. Substituting the assumed quadratic polynomial approximations into (4.165) and equating the coefficients of the different terms in the polynomials on both sides, one obtains a system of algebraic equations for the coefficients of the polynomials. Solving these equations, we obtain a first approximation to the center manifold $\boldsymbol{y}_s = \boldsymbol{h}(y_1;\mu)$. Finally, substituting this approximation into (4.160), we obtain the one-dimensional dynamical system

$$\dot{y}_1 = \hat{b}_1\mu + \xi_1 y_1\mu + \sum_{i=2}^{n} \xi_i y_i\mu + f_2[y_1, \boldsymbol{h}(y_1)] + f_3[y_1, \boldsymbol{h}(y_1)] \tag{4.166}$$

describing the dynamics of the system (4.82) on the center manifold. Next, we consider two examples.

Example 4.13

To reduce the algebra, we consider a special case of Example 4.11, namely

$$\dot{x}_1 = b_1\mu + 2x_1 - 2x_2 + \mu x_1 + \alpha_1 x_1^2 \tag{4.167}$$

$$\dot{x}_2 = b_2\mu + 4x_1 - 4x_2 - 4\mu x_2 + \alpha_2 x_1 x_2 \tag{4.168}$$

The coefficient matrix of this system at $(0, 0, 0)$ is given by (4.114). Its eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = -2$ and their corresponding eigenvectors are

$$\boldsymbol{p}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{p}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Hence,

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

Next, we introduce the transformation

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

into (4.167) and (4.168) and obtain

$$\begin{aligned} \dot{y}_1 = {} & (2b_1 - b_2)\mu + 6y_1\mu + 10y_2\mu + (2\alpha_1 - \alpha_2)y_1^2 \\ & + (4\alpha_1 - 3\alpha_2)y_1y_2 + 2(\alpha_1 - \alpha_2)y_2^2 \end{aligned} \tag{4.169}$$

$$\begin{aligned} \dot{y}_2 = {} & -2y_2 + (b_2 - b_1)\mu - 5y_1\mu - 9y_2\mu \\ & - (\alpha_1 - \alpha_2)y_1^2 - (2\alpha_1 - 3\alpha_2)y_1y_2 - (\alpha_1 - 2\alpha_2)y_2^2 \end{aligned} \tag{4.170}$$

We seek the center manifold of (4.169) and (4.170) in the form $y_2(t;\mu) = h(y_1;\mu)$ and rewrite (4.169) as

$$\dot{y}_1 = (2b_1 - b_2)\mu + 6y_1\mu + 10h\mu + (2\alpha_1 - \alpha_2)y_1^2 \\ + (4\alpha_1 - 3\alpha_2)y_1 h + 2(\alpha_1 - \alpha_2)h^2 \tag{4.171}$$

Substituting (4.171) and the expression for the center manifold into (4.170) yields

$$h'\left[(2b_1 - b_2)\mu + 6y_1\mu + 10h\mu + (2\alpha_1 - \alpha_2)y_1^2\right] = -2h + (b_2 - b_1)\mu \\ - 5y_1\mu - 9h\mu - (\alpha_1 - \alpha_2)y_1^2 - (2\alpha_1 - 3\alpha_2)y_1 h - (\alpha_1 - 2\alpha_2)h^2 + \cdots \tag{4.172}$$

whose approximate solution can be expressed as

$$h = \tfrac{1}{2}(b_2 - b_1)\mu - \tfrac{1}{2}(\alpha_1 - \alpha_2)y_1^2 + \cdots \tag{4.173}$$

Substituting (4.173) into (4.171) yields the following one-dimensional equation describing the dynamics on the center manifold:

$$\dot{y}_1 = (2b_1 - b_2)\mu + \left[5(b_2 - b_1) + \tfrac{1}{4}(\alpha_1 - \alpha_2)(b_2 - b_1)^2\right]\mu^2 + (2\alpha_1 - \alpha_2)y_1^2 \\ + \left[6 + \tfrac{1}{2}(b_2 - b_1)(4\alpha_1 - 3\alpha_2)\right] y_1\mu - \tfrac{1}{2}(\alpha_1 - \alpha_2)(4\alpha_1 - 3\alpha_2)y_1^3 \tag{4.174}$$

There are two possibilities: (a) $b_2 \neq 2b_1$ and (b) $b_2 = 2b_1$. In the first case, as $y_1 \to 0$ and $\mu \to 0$, (4.174) tends to

$$\dot{y}_1 = (2b_1 - b_2)\mu + (2\alpha_1 - \alpha_2)y_1^2 \tag{4.175}$$

Therefore, the origin of (4.167) and (4.168) undergoes a saddle-node bifurcation as μ passes through zero when $\alpha_2 \neq 2\alpha_1$. When $\alpha_2 = 2\alpha_1$, the origin of (4.167) and (4.168) is not a bifurcation point.

When $b_2 = 2b_1$, as $y_1 \to 0$ and $\mu \to 0$, (4.174) tends to

$$\dot{y}_1 = \left[5b_1 + \frac{1}{4}(\alpha_1 - \alpha_2)b_1^2\right]\mu^2 + \left[6 + \frac{1}{2}b_1(4\alpha_1 - 3\alpha_2)\right] y_1\mu + (2\alpha_1 - \alpha_2)y_1^2 \tag{4.176}$$

and therefore the origin of (4.167) and (4.168) undergoes a transcritical bifurcation as μ passes through zero when $\alpha_2 \neq 2\alpha_1$. When $\alpha_2 = 2\alpha_1$, as $y_1 \to 0$ and $\mu \to 0$, (4.174) tends to

$$\dot{y}_1 = (6 - \alpha_1 b_1)\, y_1\mu - \alpha_1^2 y_1^3 \tag{4.177}$$

and therefore the origin of (4.167) and (4.168) undergoes a supercritical pitchfork bifurcation as μ passes through zero.

Example 4.14

We reconsider the system (4.130)–(4.132) discussed in Example 4.12. In this case, the coefficient matrix of the linear system at (0, 0, 0; 0) is given by (4.142). Its eigenvalues are $\lambda = 0$ and $\lambda = 3/2\left(-1 \pm i\sqrt{3}\right)$ and their corresponding eigenvectors can be expressed as

$$\boldsymbol{p}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \boldsymbol{p}_2 = \begin{bmatrix} \frac{1}{2}\left(1 + i\sqrt{3}\right) \\ \frac{1}{2}i\left(i + \sqrt{3}\right) \\ 1 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{p}_3 = \begin{bmatrix} \frac{1}{2}\left(1 - i\sqrt{3}\right) \\ -\frac{1}{2}i\left(-i + \sqrt{3}\right) \\ 1 \end{bmatrix}$$

We note that $\boldsymbol{p}_3 = \bar{\boldsymbol{p}}_2$. Then,

$$P = \begin{bmatrix} -1 & \frac{1}{2}\left(1 + i\sqrt{3}\right) & \frac{1}{2}\left(1 - i\sqrt{3}\right) \\ 1 & \frac{1}{2}i\left(i + \sqrt{3}\right) & -\frac{1}{2}i\left(-i + \sqrt{3}\right) \\ 1 & 1 & 1 \end{bmatrix}$$

and

$$P^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6}\left(1 - i\sqrt{3}\right) & -\frac{1}{6}i\left(-i + \sqrt{3}\right) & \frac{1}{3} \\ \frac{1}{6}\left(1 + i\sqrt{3}\right) & \frac{1}{6}i\left(i + \sqrt{3}\right) & \frac{1}{3} \end{bmatrix}$$

Next, we introduce the transformation $\boldsymbol{x} = P\boldsymbol{y}$ into (4.130)–(4.132) and after some algebraic manipulations obtain

$$\begin{aligned} \dot{y}_1 = &-\tfrac{1}{3}\left(b_1 - b_2 - b_3\right)\mu - \tfrac{1}{3}\left(4\alpha_1 - \alpha_2 - 4\alpha_3\right)y_1^2 - \mu(y_2 + \bar{y}_2) \\ &- \Big[\tfrac{2}{3}\left(\alpha_1 + 3i\sqrt{3}\alpha_1 + 2\alpha_2 - \alpha_3 + 3i\sqrt{3}\alpha_3\right)y_1y_2 \\ &- \tfrac{1}{6}\left(13\alpha_1 - 3i\sqrt{3}\alpha_1 + 8\alpha_2 - 13\alpha_3 - 3i\sqrt{3}\alpha_3\right)y_2^2 \\ &+ \tfrac{1}{3}\left(7\alpha_1 - 4\alpha_2 - 7\alpha_3\right)y_2\bar{y}_2 + \text{cc}\Big] \end{aligned} \tag{4.178}$$

$$\begin{aligned} \dot{y}_2 = &-\tfrac{3}{2}\left(1 - i\sqrt{3}\right)y_2 + \tfrac{1}{6}\left(b_1 - i\sqrt{3}b_1 - b_2 - i\sqrt{3}b_2 + 2b_3\right)\mu \\ &- (y_1 + \bar{y}_2)\mu + \tfrac{1}{6}\left(4\alpha_1 - 4i\sqrt{3}\alpha_1 - \alpha_2 - i\sqrt{3}\alpha_2 + 8\alpha_3\right)y_1^2 \\ &+ \tfrac{2}{3}\left(5\alpha_1 + i\sqrt{3}\alpha_1 + \alpha_2 + i\sqrt{3}\alpha_2 + \alpha_3 - 3i\sqrt{3}\alpha_3\right)y_1y_2 \\ &+ \tfrac{2}{3}\left(-4\alpha_1 - 2i\sqrt{3}\alpha_1 + \alpha_2 + i\sqrt{3}\alpha_2 + \alpha_3 + 3i\sqrt{3}\alpha_3\right)y_1\bar{y}_2 \\ &+ \tfrac{1}{6}i\left(2i\alpha_1 + 8\sqrt{3}\alpha_1 + 4i\alpha_2 - 4\sqrt{3}\alpha_2 + 13i\alpha_3 - 3\sqrt{3}\alpha_3\right)y_2^2 \\ &+ \tfrac{1}{3}\left(7\alpha_1 - 7i\sqrt{3}\alpha_1 - 4\alpha_2 - 4i\sqrt{3}\alpha_2 + 14\alpha_3\right)y_2\bar{y}_2 \\ &+ \tfrac{1}{6}i\left(11i\alpha_1 + 5\sqrt{3}\alpha_1 + 4i\alpha_2 - 4\sqrt{3}\alpha_2 + 13i\alpha_3 + 3\sqrt{3}\alpha_3\right)\bar{y}_2^2 \end{aligned} \tag{4.179}$$

where $y_3 = \bar{y}_2$. We note that the equation governing y_3 is the complex conjugate of (4.179).

We seek the center manifold of (4.178) and (4.179) in the form $y_2(t;\mu) = h(y_1;\mu)$, where h is a complex-valued function, and rewrite (4.178) as

$$\begin{aligned}\dot{y}_1 = & -\tfrac{1}{3}(b_1 - b_2 - b_3)\mu - \tfrac{1}{3}(4\alpha_1 - \alpha_2 - 4\alpha_3)y_1^2 - \mu h - \mu\bar{h} \\ & -\Big[\tfrac{2}{3}\left(\alpha_1 + 3i\sqrt{3}\alpha_1 + 2\alpha_2 - \alpha_3 + 3i\sqrt{3}\alpha_3\right)y_1 h \\ & + \tfrac{1}{3}(7\alpha_1 - 4\alpha_2 - 7\alpha_3)h\bar{h} \\ & -\tfrac{1}{6}\left(13\alpha_1 - 3i\sqrt{3}\alpha_1 + 8\alpha_2 - 13\alpha_3 - 3i\sqrt{3}\alpha_3\right)h^2 + \text{cc}\Big]\end{aligned} \tag{4.180}$$

Substituting $y_2(t;\mu) = h(y_1;\mu)$ into (4.179) and using (4.180), we obtain

$$\begin{aligned} & h'\left[-\tfrac{1}{3}(b_1 - b_2 - b_3)\mu - \tfrac{1}{3}(4\alpha_1 - \alpha_2 - 4\alpha_3)y_1^2 - \mu h - \mu\bar{h}\right] \\ & = -\tfrac{3}{2}\left(1 - i\sqrt{3}\right)h + \tfrac{1}{6}\left(b_1 - i\sqrt{3}b_1 - b_2 - i\sqrt{3}b_2 + 2b_3\right)\mu \\ & \quad + \tfrac{1}{6}\left(4\alpha_1 - 4i\sqrt{3}\alpha_1 - \alpha_2 - i\sqrt{3}\alpha_2 + 8\alpha_3\right)y_1^2 + \cdots\end{aligned} \tag{4.181}$$

An approximate solution of (4.181) can be expressed as

$$\begin{aligned} h = & \tfrac{1}{18}\left(2b_1 + b_2 - i\sqrt{3}b_2 + b_3 + i\sqrt{3}b_3\right)\mu \\ & + \tfrac{1}{18}\left(8\alpha_1 + \alpha_2 - i\sqrt{3}\alpha_2 + 4\alpha_3 + 4i\sqrt{3}\alpha_3\right)y_1^2 + \cdots\end{aligned} \tag{4.182}$$

Substituting (4.182) into (4.180) yields the following one-dimensional equation describing the dynamics on the center manifold:

$$\begin{aligned}\dot{y}_1 = & -\tfrac{1}{3}(b_1 - b_2 - b_3)\mu - \tfrac{1}{3}(4\alpha_1 - \alpha_2 - 4\alpha_3)y_1^2 \\ & - \tfrac{4}{27}[b_3(-4\alpha_1 + \alpha_2 - 5\alpha_3) \\ & + b_1(\alpha_1 + 2\alpha_2 - \alpha_3) + b_2(5\alpha_1 + \alpha_2 + 4\alpha_3)]\mu y_1 + \Gamma_0\mu^2 \\ & - \tfrac{4}{27}\left(4\alpha_1^2 + \alpha_2^2 + \alpha_1(13\alpha_2 - 20\alpha_3) + 8\alpha_2\alpha_3 - 20\alpha_3^2\right)y_1^3\end{aligned} \tag{4.183}$$

where

$$\begin{aligned}\Gamma_0 = & \tfrac{1}{243}\big[b_2(-27 + 8b_3(5\alpha_1 + \alpha_2 - 5\alpha_3)) + b_1^2(-\alpha_1 + 16\alpha_2 + \alpha_3) \\ & - 2b_1(27 + b_2(5\alpha_1 - 8\alpha_2 + 4\alpha_3) - b_3(4\alpha_1 + 8\alpha_2 + 5\alpha_3)) \\ & + b_2^2(-25\alpha_1 + 4(\alpha_2 + 4\alpha_3)) \\ & + b_3(-27 + b_3(-16\alpha_1 + 4\alpha_2 + 25\alpha_3))\big]\end{aligned}$$

There are two possibilities: (a) $b_1 \neq b_2 + b_3$ and (b) $b_1 = b_2 + b_3$. In the first case, as $y_1 \to 0$ and $\mu \to 0$, (4.183) tends to

$$\dot{y}_1 = -\tfrac{1}{3}(b_1 - b_2 - b_3)\mu - \tfrac{1}{3}(4\alpha_1 - \alpha_2 - 4\alpha_3)y_1^2 \tag{4.184}$$

Therefore, the origin of (4.130)–(4.132) undergoes a saddle-node bifurcation as μ passes through zero when $4\alpha_1 \neq \alpha_2 + 4\alpha_3$. When $4\alpha_1 = \alpha_2 + 4\alpha_3$, the origin of (4.130)–(4.132) is not a bifurcation point. Equation 4.184 is in full agreement with (4.147) obtained with the method of multiple scales.

When $b_1 = b_2 + b_3$, as $y_1 \to 0$ and $\mu \to 0$, (4.183) tends to

$$\dot{y}_1 = \Gamma_0\mu^2 - \tfrac{4}{9}\left[b_3\left(-\alpha_1 + \alpha_2 - 2\alpha_3\right) + b_2\left(2\alpha_1 + \alpha_2 + \alpha_3\right)\right]\mu y_1 - \tfrac{1}{3}\left(4\alpha_1 - \alpha_2 - 4\alpha_3\right)y_1^2 \tag{4.185}$$

where

$$\Gamma_0 = \tfrac{1}{27}b_2\left[-9 + 4b_3\left(\alpha_1 + 2\alpha_2 - \alpha_3\right)\right] + \tfrac{1}{27}b_2^2\left(-4\alpha_1 + 4\alpha_2 + \alpha_3\right) + \tfrac{1}{27}b_3\left[-9 + b_3\left(-\alpha_1 + 4\left(\alpha_2 + \alpha_3\right)\right)\right]$$

Therefore, the origin of (4.130)–(4.132) undergoes a transcritical bifurcation as μ passes through zero when $4\alpha_1 \neq \alpha_2 + 4\alpha_3$. Equation 4.185 is in full agreement with (4.152) obtained with the method of multiple scales.

When $4\alpha_1 = \alpha_2 + 4\alpha_3$, as $y_1 \to 0$ and $\mu \to 0$, (4.183) tends to

$$\dot{y}_1 = \tfrac{1}{3}\left[b_3\left(4\alpha_3 - \alpha_2\right) - 2b_2\left(\alpha_2 + 2\alpha_3\right)\right]\mu y_1 - \tfrac{2}{3}\left(\alpha_2^2 + 4\alpha_2\alpha_3 - 8\alpha_3^2\right)y_1^3 \tag{4.186}$$

Therefore, the origin of (4.130)–(4.132) undergoes a supercritical or subcritical pitchfork bifurcation as μ passes through zero, depending on whether $\alpha_2^2 + 4\alpha_2\alpha_3 - 8\alpha_3^2$ is positive or negative, respectively. Equation 4.186 is in full agreement with (4.157) obtained with the method of multiple scales.

4.5.3
A Projection Method

In this section, we propose a method of reducing (4.82) directly without calculating the center manifold. We decompose its solution as follows:

$$\boldsymbol{x} = \boldsymbol{p}u(t) + \boldsymbol{y}(t) \tag{4.187}$$

where $\boldsymbol{p}$ is the right eigenvector of A,

$$\boldsymbol{q}^{\mathrm{T}}\boldsymbol{y}(t) = 0 \tag{4.188}$$

and $\boldsymbol{q}$ is the left eigenvector of A normalized so that $\boldsymbol{q}^{\mathrm{T}}\boldsymbol{p} = 1$. Substituting (4.187) into (4.82) yields

$$\boldsymbol{p}\dot{u} + \dot{\boldsymbol{y}} = A\boldsymbol{y} + \boldsymbol{b}\mu + B\boldsymbol{p}\mu u + B\boldsymbol{y}\mu + \boldsymbol{Q}(\boldsymbol{p},\boldsymbol{p})u^2 + 2\boldsymbol{Q}(\boldsymbol{p},\boldsymbol{y})u + \boldsymbol{Q}(\boldsymbol{y},\boldsymbol{y}) + \boldsymbol{C}(\boldsymbol{p},\boldsymbol{p},\boldsymbol{p})u^3 + \cdots \tag{4.189}$$

Multiplying (4.189) from the left with $\boldsymbol{q}$ and using (4.188) leads to

$$\dot{u} = \boldsymbol{q}^{\mathrm{T}}\boldsymbol{b}\mu + \boldsymbol{q}^{\mathrm{T}}B\boldsymbol{p}\mu u + \boldsymbol{q}^{\mathrm{T}}B\boldsymbol{y}\mu + \boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{p},\boldsymbol{p})u^2 + 2\boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{p},\boldsymbol{y})u + \boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{y},\boldsymbol{y}) + \boldsymbol{q}^{\mathrm{T}}\boldsymbol{C}(\boldsymbol{p},\boldsymbol{p},\boldsymbol{p})u^3 + \cdots \tag{4.190}$$

Substituting for $\dot{u}$ from (4.190) and (4.189), we obtain

$$\dot{\boldsymbol{y}} = A\boldsymbol{y} + \boldsymbol{b}\mu - (\boldsymbol{q}^{\mathrm{T}}\boldsymbol{b})\boldsymbol{p}\mu + \boldsymbol{Q}(\boldsymbol{p},\boldsymbol{p})u^2 - \left(\boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{p},\boldsymbol{p})\right)\boldsymbol{p}u^2 + \cdots \tag{4.191}$$

There are two possibilities: $\boldsymbol{q}^{\mathrm{T}}\boldsymbol{b} \neq 0$ and $\boldsymbol{q}^{\mathrm{T}}\boldsymbol{b} = 0$. In the first case, as $u \to 0$ and $\mu \to 0$, (4.190) tends to

$$\dot{u} = \boldsymbol{q}^{\mathrm{T}}\boldsymbol{b}\mu + \boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{p},\boldsymbol{p})u^2 \tag{4.192}$$

and therefore the origin of (4.82) undergoes a generic saddle-node bifurcation as μ passes through zero when $\boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{p},\boldsymbol{p}) \neq 0$.

When $\boldsymbol{q}^{\mathrm{T}}\boldsymbol{b} = 0$, we express the solution of (4.191) and (4.188) as

$$\boldsymbol{y} = \boldsymbol{c}\mu + \boldsymbol{z}u^2 \tag{4.193}$$

where

$$A\boldsymbol{c} = -\boldsymbol{b} \quad \text{and} \quad A\boldsymbol{z} = -\boldsymbol{Q}(\boldsymbol{p},\boldsymbol{p}) + \left(\boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{p},\boldsymbol{p})\right)\boldsymbol{p} \tag{4.194}$$

and

$$\boldsymbol{q}^{\mathrm{T}}\boldsymbol{c} = 0 \quad \text{and} \quad \boldsymbol{q}^{\mathrm{T}}\boldsymbol{z} = 0 \tag{4.195}$$

Substituting (4.193) into (4.190), we find that, as $u \to 0$ and $\mu \to 0$, (4.190) tends to

$$\dot{u} = \left[\boldsymbol{q}^{\mathrm{T}}B\boldsymbol{c} + \boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{c},\boldsymbol{c})\right]\mu^2 + \boldsymbol{q}^{\mathrm{T}}B\boldsymbol{p}\mu u + \boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{p},\boldsymbol{p})u^2 \tag{4.196}$$

when $\boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{p},\boldsymbol{p}) \neq 0$. Therefore, the origin of (4.82) undergoes a generic transcritical bifurcation as μ passes through zero.

When $\boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{p},\boldsymbol{p}) = 0$, as $u \to 0$ and $\mu \to 0$, (4.190) tends to

$$\dot{u} = \boldsymbol{q}^{\mathrm{T}}B\boldsymbol{p}\mu u + \left[2\boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{p},\boldsymbol{z}) + \boldsymbol{q}^{\mathrm{T}}\boldsymbol{C}(\boldsymbol{p},\boldsymbol{p},\boldsymbol{p})\right]u^3 \tag{4.197}$$

Therefore, the origin of (4.82) undergoes a generic supercritical or subcritical pitchfork bifurcation as μ passes through zero depending on whether $[2\boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{p},\boldsymbol{z}) + \boldsymbol{q}^{\mathrm{T}}\boldsymbol{C}(\boldsymbol{p},\boldsymbol{p},\boldsymbol{p})]$ is negative or positive.

Example 4.15

We reconsider (4.167) and (4.168). Their Jacobian matrix evaluated at $(0,0,0)$ is given in (4.114) and its eigenvalues are $\lambda = 0$ and $\lambda = -2$. The right and left eigenvectors of A are

$$\boldsymbol{p} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{q} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Therefore, we express the solution of (4.167) and (4.168) as

$$\boldsymbol{x} = \boldsymbol{p}u(t) + \boldsymbol{y}(t)$$

or

$$x_1 = u(t) + y_1(t) \text{ and } x_2 = u(t) + y_2(t) \tag{4.198}$$

and impose the condition $\boldsymbol{q}^{\mathrm{T}}\boldsymbol{y} = 0$. Substituting (4.198) into (4.167) and (4.168) yields

$$\dot{u} + \dot{y}_1 = b_1\mu + 2y_1 - 2y_2 + \mu(u + y_1) + \alpha_1(u + y_1)^2 \tag{4.199}$$

$$\dot{u} + \dot{y}_2 = b_2\mu + 4y_1 - 4y_2 - 4(u + y_2)\mu + (u + y_1)(u + y_2) \tag{4.200}$$

The condition $\boldsymbol{q}^{\mathrm{T}}\boldsymbol{y} = 0$ yields

$$2y_1 - y_2 = 0 \tag{4.201}$$

Multiplying (4.199) and (4.200) from the left with $\boldsymbol{q}^{\mathrm{T}}$ and using the condition $\boldsymbol{q}^{\mathrm{T}}\boldsymbol{y} = 0$, we obtain

$$\begin{aligned}\dot{u} = {} & (2b_1 - b_2)\mu + (2\alpha_1 - \alpha_2)u^2 + 6\mu u + 2\mu y_1 + 4\mu y_2 \\ & + (4\alpha_1 - \alpha_2)uy_1 - \alpha_2 uy_2 + 2\alpha_1 y_1^2 - \alpha_2 y_1 y_2\end{aligned} \tag{4.202}$$

Substituting (4.202) into (4.199) and (4.200) yields

$$\dot{y}_1 = (b_2 - b_1)\mu + 2y_1 - 2y_2 + (\alpha_2 - \alpha_1)u^2 + \cdots \tag{4.203}$$

$$\dot{y}_2 = 2(b_2 - b_1)\mu + 4y_1 - 4y_2 + 2(\alpha_2 - \alpha_1)u^2 + \cdots \tag{4.204}$$

Because $u(t)$ varies slowly with time, the solution of (4.204) and (4.201) can be expressed as

$$y_1 = \tfrac{1}{2}(b_2 - b_1)\mu + \tfrac{1}{2}(\alpha_2 - \alpha_1)u^2 + \cdots \quad \text{and} \quad y_2 = 2y_1 \tag{4.205}$$

It follows from (4.202) that there are two possibilities: $b_2 \neq 2b_1$ and $b_2 = 2b_1$. In the first case, as $u \to 0$ and $\mu \to 0$, (4.202) tends to

$$\dot{u} = (2b_1 - b_2)\mu + (2\alpha_1 - \alpha_2)u^2 \tag{4.206}$$

Therefore, the origin of (4.167) and (4.168) undergoes a saddle-node bifurcation as μ passes through zero if $\alpha_2 \neq 2\alpha_1$. We note that (4.206) is in full agreement with (4.118) and (4.175) obtained with the methods of multiple scales and center-manifold reduction, respectively.

When $b_2 = 2b_1$, substituting (4.205) into (4.202) yields

$$\dot{u} = \tfrac{1}{2} b_1 (10 + b_1 \alpha_1 - b_1 \alpha_2)\mu^2 + \tfrac{1}{2}(12 + 4b_1\alpha_1 - 3b_1\alpha_2)\mu u + (2\alpha_1 - \alpha_2)u^2 - \tfrac{1}{2}(4\alpha_1 - 3\alpha_2)(\alpha_1 - \alpha_2)u^3 + \cdots \tag{4.207}$$

As $u \to 0$ and $\mu \to 0$, (4.207) tends to

$$\dot{u} = \tfrac{1}{2} b_1 (10 + b_1 \alpha_1 - b_1 \alpha_2)\mu^2 + \tfrac{1}{2}(12 + 4b_1\alpha_1 - 3b_1\alpha_2)\mu u + (2\alpha_1 - \alpha_2)u^2 \tag{4.208}$$

Therefore, the origin of (4.167) and (4.168) undergoes a transcritical bifurcation as μ passes through zero if $\alpha_2 \neq 2\alpha_1$. We note that (4.208) is in full agreement with (4.122) and (4.176) obtained with the methods of multiple scales and center-manifold reduction, respectively.

When $\alpha_2 = 2\alpha_1$, as $u \to 0$ and $\mu \to 0$, (4.207) tends to

$$\dot{u} = (6 - b_1\alpha_1)\mu u - \alpha_1^2 u^3 \tag{4.209}$$

Therefore, the origin of (4.167) and (4.168) undergoes a supercritical pitchfork bifurcation as μ passes through zero if $\alpha_2 = 2\alpha_1$. We note that (4.209) is in full agreement with (4.129) and (4.177) obtained with the methods of multiple scales and center-manifold reduction, respectively.

Example 4.16

We reconsider the system (4.130)–(4.132) treated in Examples 4.12 and 4.14. The coefficient matrix of this system evaluated at (0, 0, 0; 0) is given by (4.142). Its eigenvalues are $\lambda = 0$ and $\lambda = 3/2(-1 \pm i\sqrt{3})$. Hence, the origin of (4.130)–(4.133) may be a static bifurcation point. The right and left eigenvectors of A corresponding to $\lambda = 0$ are

$$\boldsymbol{p} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{q} = \frac{1}{3}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

where $\boldsymbol{q}$ is normalized so that $\boldsymbol{q}^{\mathrm{T}}\boldsymbol{p} = 1$.

We express the solution of (4.130)–(4.132) as

$$\boldsymbol{x} = \boldsymbol{p}\,u(t) + \boldsymbol{y}(t) \quad \text{where} \quad \boldsymbol{q}^{\mathrm{T}}\boldsymbol{y} = 0$$

or

$$x_1 = -u + y_1\,, \quad x_2 = u + y_2\,, \quad x_3 = u + y_3 \tag{4.210}$$

$$y_1 - y_2 - y_3 = 0 \tag{4.211}$$

Substituting (4.210) into (4.130)–(4.132), we have

$$\dot{y}_1 - \dot{u} = b_1\mu - y_1 + y_2 - 2y_3 - \mu u + \mu y_1 + \alpha_1 (2u + y_1 + 2y_2 + y_3)^2 \tag{4.212}$$

$$\dot{y}_2 + \dot{u} = b_2\mu - 2y_1 - y_2 - y_3 + \mu u + \mu y_2 + \alpha_2 (-u + y_1 - y_2 + y_3)^2 \tag{4.213}$$

$$\dot{y}_3 + \dot{u} = b_3\mu + y_1 + 2y_2 - y_3 - 2\mu u - 2\mu y_3 + \alpha_3 (-2u + 2y_1 + y_2 - y_3)^2 \tag{4.214}$$

Multiplying (4.212)–(4.213) from the left with $\boldsymbol{q}^{\mathrm{T}}$ yields

$$\begin{aligned}\dot{u} = &-\tfrac{1}{3}(b_1 - b_2 - b_3)\mu + \tfrac{1}{3}(4\alpha_3 + \alpha_2 - 4\alpha_1)u^2 - (y_1 - y_2 + 2y_3)\mu \\ &- \tfrac{2}{3}(4\alpha_3 + \alpha_2 + 2\alpha_1)uy_1 - \tfrac{2}{3}(2\alpha_3 - \alpha_2 + 4\alpha_1)uy_2 \\ &+ \tfrac{2}{3}(2\alpha_3 - \alpha_2 - 2\alpha_1)uy_3 + \tfrac{1}{3}(4\alpha_3 + \alpha_2 - \alpha_1)y_1^2 \\ &+ \tfrac{1}{3}(\alpha_3 + \alpha_2 - 4\alpha_1)y_2^2 + \tfrac{1}{3}(\alpha_3 + \alpha_2 - 4\alpha_1)y_2^2 \\ &+ \tfrac{1}{3}(\alpha_3 + \alpha_2 - \alpha_1)y_3^2 + \tfrac{2}{3}(2\alpha_3 - \alpha_2 - 2\alpha_1)y_1y_2 \\ &- \tfrac{2}{3}(2\alpha_3 - \alpha_2 + \alpha_1)y_1y_3 - \tfrac{2}{3}(\alpha_3 + \alpha_2 + 2\alpha_1)y_2y_3\end{aligned} \tag{4.215}$$

Substituting (4.215) into (4.212)–(4.213) yields

$$\dot{y}_1 = \tfrac{1}{3}(4b_1 - b_2 - b_3)\mu - y_1 + y_2 - 2y_3 + \tfrac{2}{3}(2\alpha_1 + \alpha_2 - 2\alpha_3)u^2 + \cdots \tag{4.216}$$

$$\dot{y}_2 = \tfrac{1}{3}(b_1 + 2b_2 - b_3)\mu - 2y_1 - y_2 - y_3 + \tfrac{1}{3}(4\alpha_1 - \alpha_2 + 8\alpha_3)u^2 + \cdots \tag{4.217}$$

$$\dot{y}_3 = \tfrac{1}{3}(b_1 - b_2 + 2b_3)\mu + y_1 + 2y_2 - y_3 + \tfrac{1}{3}(4\alpha_1 - \alpha_2 + 8\alpha_3)u^2 + \cdots \tag{4.218}$$

Because $u(t)$ is a slowly varying function of time, the particular solution of (4.216)–(4.218) orthogonal to $\boldsymbol{q}$ can be expressed as

$$y_1 = \tfrac{1}{9}(b_1 + 2b_2 - b_3)\mu + \tfrac{2}{9}(2\alpha_1 + \alpha_2 - 2\alpha_3)u^2 + \cdots \tag{4.219}$$

$$y_2 = \tfrac{1}{9}(b_2 - b_1 - 2b_3)\mu - \tfrac{1}{9}(4\alpha_1 - \alpha_2 + 8\alpha_3)u^2 + \cdots \tag{4.220}$$

$$y_3 = \tfrac{1}{9}(2b_1 + b_2 + b_3)\mu + \tfrac{1}{9}(8\alpha_1 + \alpha_2 + 4\alpha_3)u^2 + \cdots \tag{4.221}$$

There are two possibilities: $b_1 - b_2 - b_3 \neq 0$ and $b_1 - b_2 - b_3 = 0$. In the first case, as $u \to 0$ and $\mu \to 0$, (4.215) tends to

$$\dot{u} = -\tfrac{1}{3}(b_1 - b_2 - b_3)\mu + \tfrac{1}{3}(4\alpha_3 + \alpha_2 - 4\alpha_1)u^2 \tag{4.222}$$

Therefore, the origin of (4.130)–(4.132) undergoes a saddle-node bifurcation as μ passes through zero when $4\alpha_1 \neq \alpha_2 + 4\alpha_3$. When $4\alpha_1 = \alpha_2 + 4\alpha_3$, the origin of (4.130)–(4.132) is not a bifurcation point. Equation 4.222 is in full agreement with (4.147) and (4.184) obtained with the methods of multiple scales and center-manifold reduction, respectively.

When $b_1 = b_2 + b_3$, substituting (4.219)–(4.221) into (4.215), we have

$$\begin{aligned}\dot{u} = &-\tfrac{1}{27}\left(9b_2 + 9b_3 + 4b_2^2\alpha_1 - 4b_2b_3\alpha_1 + b_3^2\alpha_1 - 4b_2^2\alpha_2 - 8b_2b_3\alpha_2\right.\\ &\left.-4b_3^2\alpha_2 - b_2^2\alpha_3 + 4b_2b_3\alpha_3 - 4b_3^2\alpha_3\right)\mu^2\\ &-\tfrac{4}{27}\left(b_1\alpha_1 + 5b_2\alpha_1 - 4b_3\alpha_1 + 2b_1\alpha_2 + b_2\alpha_2\right.\\ &\left.+b_2\alpha_2 + b_3\alpha_2 - b_1\alpha_3 + 4b_2\alpha_3 - 5b_3\alpha_3\right)\mu u + \tfrac{1}{3}(4\alpha_3 + \alpha_2 - 4\alpha_1)u^2\\ &-\tfrac{4}{27}\left(4\alpha_1^2 + 13\alpha_1\alpha_2 + \alpha_2^2 - 20\alpha_1\alpha_3 + 8\alpha_2\alpha_3 - 20\alpha_3^2\right)u^3\end{aligned} \tag{4.223}$$

When $4\alpha_1 \neq \alpha_2 + 4\alpha_3$, as $u \to 0$ and $\mu \to 0$, (4.223) tends to

$$\begin{aligned}\dot{u} = &-\tfrac{1}{27}\left(9b_2 + 9b_3 + 4b_2^2\alpha_1 - 4b_2b_3\alpha_1 + b_3^2\alpha_1 - 4b_2^2\alpha_2 - 8b_2b_3\alpha_2\right.\\ &\left.-4b_3^2\alpha_2 - b_2^2\alpha_3 + 4b_2b_3\alpha_3 - 4b_3^2\alpha_3\right)\mu^2\\ &-\tfrac{4}{27}\left(b_1\alpha_1 + 5b_2\alpha_1 - 4b_3\alpha_1 + 2b_1\alpha_2 + b_2\alpha_2\right.\\ &\left.+b_2\alpha_2 + b_3\alpha_2 - b_1\alpha_3 + 4b_2\alpha_3 - 5b_3\alpha_3\right)\mu u\\ &+\tfrac{1}{3}(4\alpha_3 + \alpha_2 - 4\alpha_1)u^2\end{aligned} \tag{4.224}$$

Therefore, the origin of (4.130)–(4.132) undergoes a transcritical bifurcation as μ passes through zero when $4\alpha_1 \neq \alpha_2 + 4\alpha_3$. Equation 4.224 is in full agreement with (4.152) and (4.185) obtained with the methods of multiple scales and center-manifold reduction, respectively.

When $\alpha_1 = \alpha_3 + 1/4\alpha_2$, as $u \to 0$ and $\mu \to 0$, (4.223) tends to

$$\dot{u} = \tfrac{1}{3}\left[b_3(4\alpha_3 - \alpha_2) - 2b_2(\alpha_2 + 2\alpha_3)\right]\mu u - \tfrac{2}{3}\left(\alpha_2^2 + 4\alpha_2\alpha_3 - 8\alpha_3^2\right)u^3 \tag{4.225}$$

Therefore, the origin of (4.130)–(4.132) undergoes a supercritical or subcritical pitchfork bifurcation as μ passes through zero, depending on whether $\alpha_2^2 + 4\alpha_2\alpha_3 - 8\alpha_3^2$ is positive or negative, respectively. Equation 4.225 is in full agreement with (4.157) and (4.186) obtained with the methods of multiple scales and center-manifold reduction, respectively.

4.6
Normal Form of Hopf Bifurcation

In this section, we describe methods for simplifying the dynamical system (4.17) near a Hopf bifurcation point. We assume that this point has been shifted to $\boldsymbol{x} = \boldsymbol{0}$ and that the control parameter has been shifted to $\mu = 0$. Moreover, we expand

(4.17) for small $\boldsymbol{x}$ and μ and obtain (4.82). Furthermore, because A is nonsingular, we can shift the fixed point by $-A^{-1}\boldsymbol{b}\mu$ and hence rewrite (4.82) as

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + \mu B\boldsymbol{x} + \boldsymbol{Q}(\boldsymbol{x}, \boldsymbol{x}) + \boldsymbol{C}(\boldsymbol{x}, \boldsymbol{x}, \boldsymbol{x}) + \cdots \tag{4.226}$$

Because the origin is a Hopf bifurcation, two of the eigenvalues are purely imaginary complex conjugates (i.e., $\pm i\omega$) and the remaining eigenvalues are in the left-half of the complex plane. Next, we demonstrate how one can use the methods of multiple scales, projection method, and center-manifold reduction to compute the normal form of (4.226) near $\boldsymbol{x} = 0$ and small μ.

4.6.1
The Method of Multiple Scales

To compute the normal form of the Hopf bifurcation of (4.226) at the origin, we introduce a small nondimensional parameter ϵ as a bookkeeping parameter and seek a third-order approximate solution of (4.226) in the form

$$\boldsymbol{x}(t;\mu) = \epsilon\boldsymbol{x}_1(T_0, T_2) + \epsilon^2\boldsymbol{x}_2(T_0, T_2) + \epsilon^3\boldsymbol{x}_3(T_0, T_2) + \cdots \tag{4.227}$$

where the time scales $T_m = \epsilon^m t$. The time derivative can be expressed in terms of these scales as

$$\frac{d}{dt} = D_0 + \epsilon^2 D_2 + \cdots \tag{4.228}$$

where $D_m = \partial/\partial T_m$. As shown below, there is no dependence on T_1 because the second-order problem is solvable. Moreover, because μ is small, we scale it as $\epsilon^2\mu$. Substituting (4.227) and (4.228) into (4.226) and equating coefficients of like powers of ϵ, we obtain

Order (ϵ)

$$D_0\boldsymbol{x}_1 - A\boldsymbol{x}_1 = \boldsymbol{0} \tag{4.229}$$

Order (ϵ^2)

$$D_0\boldsymbol{x}_2 - A\boldsymbol{x}_2 = \boldsymbol{Q}(\boldsymbol{x}_1, \boldsymbol{x}_1) \tag{4.230}$$

Order (ϵ^3)

$$D_0\boldsymbol{x}_3 - A\boldsymbol{x}_3 = -D_2\boldsymbol{x}_1 + \mu B\boldsymbol{x}_1 + 2\boldsymbol{Q}(\boldsymbol{x}_1, \boldsymbol{x}_2) + \boldsymbol{C}(\boldsymbol{x}_1, \boldsymbol{x}_1, \boldsymbol{x}_1) \tag{4.231}$$

The general solution of (4.229) is the superposition of n linearly independent solutions corresponding to the n eigenvalues of A: $n-2$ of these eigenvalues have negative real parts and two are purely imaginary. The $n-2$ solutions corresponding to the eigenvalues with negative real parts decay with time and hence the long-time dynamics of the system depends on the center space corresponding to the two purely imaginary eigenvalues. Therefore, we express the solution of (4.229) as

$$x_1 = z(T_2)\boldsymbol{p}e^{i\omega T_0} + \bar{z}(T_2)\bar{\boldsymbol{p}}e^{-i\omega T_0} \tag{4.232}$$

where the function $z(T_2)$ is determined by imposing the solvability condition at third order and $\boldsymbol{p}$ is the eigenvector of A corresponding to the eigenvalue $i\omega$; that is,

$$A\boldsymbol{p} = i\omega\boldsymbol{p} \tag{4.233}$$

Substituting (4.232) into (4.230) yields

$$D_0\boldsymbol{x}_2 - A\boldsymbol{x}_2 = \boldsymbol{Q}(\boldsymbol{p},\boldsymbol{p})z^2e^{2i\omega T_0} + 2\boldsymbol{Q}(\boldsymbol{p},\bar{\boldsymbol{p}})z\bar{z} + \boldsymbol{Q}(\bar{\boldsymbol{p}},\bar{\boldsymbol{p}})\bar{z}^2e^{-2i\omega T_0} \tag{4.234}$$

The solution of (4.234) can be expressed as

$$\boldsymbol{x}_2 = \boldsymbol{\zeta}_2 z^2 e^{2i\omega T_0} + 2\boldsymbol{\zeta}_0 z\bar{z} + \bar{\boldsymbol{\zeta}}_2\bar{z}^2e^{-2i\omega T_0} \tag{4.235}$$

where

$$[2i\omega I - A]\boldsymbol{\zeta}_2 = \boldsymbol{Q}(\boldsymbol{p},\boldsymbol{p}) \quad \text{and} \quad A\boldsymbol{\zeta}_0 = -\boldsymbol{Q}(\boldsymbol{p},\bar{\boldsymbol{p}}) \tag{4.236}$$

Substituting (4.232) and (4.235) into (4.231), we have

$$\begin{aligned} D_0\boldsymbol{x}_3 - A\boldsymbol{x}_3 = -\big[&D_2 z\boldsymbol{p} - \mu B\boldsymbol{p}z - 4\boldsymbol{Q}(\boldsymbol{p},\boldsymbol{\zeta}_0)z^2\bar{z} - 2\boldsymbol{Q}(\bar{\boldsymbol{p}},\boldsymbol{\zeta}_2)z^2\bar{z} \\ &- 3\boldsymbol{C}(\boldsymbol{p},\boldsymbol{p},\bar{\boldsymbol{p}})z^2\bar{z}\big]e^{i\omega T_0} + \text{cc} + \text{NST}\,. \end{aligned} \tag{4.237}$$

Because $\boldsymbol{p}e^{i\omega T_0}$ is a solution of the homogeneous equation 4.237, the nonhomogeneous equation has a solution only if a solvability condition is satisfied. We let $\boldsymbol{q}$ be the left eigenvector of A corresponding to the eigenvalue $i\omega$; that is,

$$A^{\mathrm{T}}\boldsymbol{q} = i\omega\boldsymbol{q} \tag{4.238}$$

We normalize it so that $\boldsymbol{q}^{\mathrm{T}}\boldsymbol{p} = 1$. Then, the solvability condition demands that the term proportional to $e^{i\omega T_0}$ in (4.237) is orthogonal to $\boldsymbol{q}$. Imposing this condition, we obtain the following normal form of the Hopf bifurcation:

$$D_2 z = \boldsymbol{q}^{\mathrm{T}}B\boldsymbol{p}\mu z + \big[4\boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{p},\boldsymbol{\zeta}_0) + 2\boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}(\bar{\boldsymbol{p}},\boldsymbol{\zeta}_2) + 3\boldsymbol{q}^{\mathrm{T}}\boldsymbol{C}(\boldsymbol{p},\boldsymbol{p},\bar{\boldsymbol{p}})\big]z^2\bar{z} \tag{4.239}$$

Example 4.17

We consider the system

$$\dot{x}_1 = \mu x_1 - x_2 + x_1^2 - \alpha_1 x_1 x_3 \tag{4.240}$$

$$\dot{x}_2 = \mu x_2 + x_1 + \alpha_2 x_2 x_3 \tag{4.241}$$

$$\dot{x}_3 = -x_3 + x_1^2 + 2x_2^2 \tag{4.242}$$

Substituting (4.227) and (4.228) into (4.240)–(4.242), scaling μ as $\epsilon^2\mu$, and equating coefficients of like powers of ϵ on both sides, we obtain

Order (ϵ)

$$D_0 x_{11} + x_{21} = 0 \tag{4.243}$$

$$D_0 x_{21} - x_{11} = 0 \tag{4.244}$$

$$D_0 x_{31} + x_{31} = 0 \tag{4.245}$$

Order (ϵ^2)

$$D_0 x_{12} + x_{22} = x_{11}^2 - \alpha_1 x_{11} x_{31} \tag{4.246}$$

$$D_0 x_{22} - x_{12} = \alpha_2 x_{21} x_{31} \tag{4.247}$$

$$D_0 x_{32} + x_{32} = x_{11}^2 + 2x_{21}^2 \tag{4.248}$$

Order (ϵ^3)

$$D_0 x_{13} + x_{23} = -D_2 x_{11} + \mu x_{11} + 2x_{11} x_{12} - \alpha_1 x_{11} x_{32} - \alpha_1 x_{12} x_{31} \tag{4.249}$$

$$D_0 x_{23} - x_{13} = -D_2 x_{21} + \mu x_{21} + \alpha_2 x_{21} x_{32} + \alpha_2 x_{22} x_{31} \tag{4.250}$$

$$D_0 x_{33} + x_{33} = -D_2 x_{31} + 2x_{11} x_{12} + 4x_{21} x_{22} \tag{4.251}$$

The nondecaying solutions of (4.243)–(4.245) can be expressed as

$$\begin{aligned} x_{11} &= iz(T_2)e^{iT_0} - i\bar{z}(T_2)e^{-iT_0}\,, \\ x_{21} &= z(T_2)e^{iT_0} + \bar{z}(T_2)e^{-iT_0}\,, \\ x_{31} &= 0 \end{aligned} \tag{4.252}$$

Substituting (4.252) into (4.246)–(4.248) yields

$$D_0 x_{12} + x_{22} = z^2 e^{2iT_0} - 2z\bar{z} + \bar{z}^2 e^{-2iT_0} \tag{4.253}$$

$$D_0 x_{22} - x_{12} = 0 \tag{4.254}$$

$$D_0 x_{32} + x_{32} = z^2 e^{2iT_0} + 6z\bar{z} + \bar{z}^2 e^{-2iT_0} \tag{4.255}$$

whose solutions can be expressed as

$$x_{12} = -\tfrac{2}{3} i z^2 e^{2iT_0} + \tfrac{2}{3} i \bar{z}^2 e^{-2iT_0} \tag{4.256}$$

$$x_{22} = -\tfrac{1}{3} z^2 e^{2iT_0} - 2z\bar{z} - \tfrac{1}{3}\bar{z}^2 e^{-2iT_0} \tag{4.257}$$

$$x_{32} = \tfrac{1}{5}(1 - 2i) z^2 e^{2iT_0} + 6z\bar{z} + \tfrac{1}{5}(1 + 2i)\bar{z}^2 e^{-2iT_0} \tag{4.258}$$

Substituting (4.252) and (4.256)–(4.258) into (4.249) and (4.250), we have

$$D_0 x_{13} + x_{23} = \left[-i D_2 z + i\mu z + \left(\tfrac{4}{3} + \tfrac{2}{5}\alpha_1(2 - 29i)\right) z^2\bar{z}\right] e^{iT_0} + \text{cc} + \text{NST} \tag{4.259}$$

$$D_0 x_{23} - x_{13} = \left[-D_2 z + \mu z + \tfrac{1}{5}\alpha_2(31 - 2i) z^2 \bar{z}\right] e^{iT_0} + \text{cc} + \text{NST} \tag{4.260}$$

The solvability condition of (4.259) and (4.260) demands that their right-hand sides be orthogonal to $\boldsymbol{q}$, where $\boldsymbol{q}^{\mathrm{T}} = 1/2(-i, 1)$. Imposing this condition, we obtain the following normal form of the system (4.240)–(4.242) near its origin:

$$D_2 z = \mu z + \tfrac{1}{30}\left[93\alpha_2 - 87\alpha_1 - (20 + 6\alpha_1 + 6\alpha_2) i\right] z^2 \bar{z} \tag{4.261}$$

Letting $z = 1/2 e^{i\theta}$ in (4.261) and separating real and imaginary parts, we obtain the following alternate normal form of the Hopf bifurcation:

$$D_2 r = \tfrac{1}{2}\mu r + \tfrac{1}{40}(31\alpha_2 - 29\alpha_1) r^3 \tag{4.262}$$

$$D_2 \theta = -\tfrac{1}{60}(10 + 3\alpha_1 + 3\alpha_2) r^2 \tag{4.263}$$

4.6.2 Center-Manifold Reduction

To determine the normal form of the Hopf bifurcation of the system (4.226) at the origin, we first calculate the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and eigenvectors (generalized eigenvectors) $\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n$ of the matrix A. Second, we arrange the eigenvalues of A so that $\lambda_1 = i\omega$ and $\lambda_2 = -i\omega$ and let $\boldsymbol{p}_1$ and $\bar{\boldsymbol{p}}_1$ be their corresponding eigenvectors. Third, we introduce the transformation $\boldsymbol{x} = P\boldsymbol{y}$ and obtain

$$\dot{\boldsymbol{y}} = J\boldsymbol{y} + P^{-1} B P \boldsymbol{y}\mu + P^{-1}\boldsymbol{Q}(P\boldsymbol{y}, P\boldsymbol{y}) + P^{-1}\boldsymbol{C}(P\boldsymbol{y}, P\boldsymbol{y}, P\boldsymbol{y}) \tag{4.264}$$

where $J = P^{-1}AP$. We note that J can be rewritten as

$$J = \begin{bmatrix} J_c & 0 \\ 0 & J_s \end{bmatrix} \quad \text{and} \quad J_c = \begin{bmatrix} i\omega & 0 \\ 0 & -i\omega \end{bmatrix} \tag{4.265}$$

and J_s is an $(n-2) \times (n-2)$ matrix whose eigenvalues are $\lambda_3, \lambda_4, \ldots, \lambda_n$. Fourth, we let $\boldsymbol{y}_c$ be the two-dimensional vector with the components y_1 and y_2 and let $\boldsymbol{y}_s$ be the $(n-2)$-dimensional vector with the components $y_3, y_4, \ldots, y_n$ and rewrite (4.264) as

$$\dot{\boldsymbol{y}}_c = J_c \boldsymbol{y}_c + \mu B_{1c} \boldsymbol{y}_c + \mu B_{1s} \boldsymbol{y}_s + \boldsymbol{G}_2(\boldsymbol{y}_c, \boldsymbol{y}_s) + \boldsymbol{G}_3(\boldsymbol{y}_c, \boldsymbol{y}_s) \tag{4.266}$$

and

$$\dot{\boldsymbol{y}}_s = J_s \boldsymbol{y}_s + \mu B_{2c} \boldsymbol{y}_c + \mu B_{2s} \boldsymbol{y}_s + \boldsymbol{F}_2(\boldsymbol{y}_c, \boldsymbol{y}_s) + \boldsymbol{F}_3(\boldsymbol{y}_c, \boldsymbol{y}_s) \tag{4.267}$$

We note that $\boldsymbol{y}_c$ and $\boldsymbol{y}_s$ are linearly uncoupled but nonlinearly coupled. Further, $\boldsymbol{F}_j(\boldsymbol{0}, \boldsymbol{0}) = \boldsymbol{0}$, $\boldsymbol{G}_j(\boldsymbol{0}, \boldsymbol{0}) = \boldsymbol{0}$, and the Jacobian matrices $D\boldsymbol{F}_j(\boldsymbol{0}, \boldsymbol{0})$ and $D\boldsymbol{G}_j(\boldsymbol{0}, \boldsymbol{0})$ are matrices with zero entries.

Because $\boldsymbol{F}_j$ and $\boldsymbol{G}_j$ are polynomials and hence infinitely differentiable, there exists a *local center manifold* of the form

$$\boldsymbol{y}_s = \boldsymbol{h}(\boldsymbol{y}_c)$$

where $\boldsymbol{h}$ is a polynomial function of $\boldsymbol{y}_c$ and

$$\boldsymbol{h}(\boldsymbol{0}) = \boldsymbol{0} \quad \text{and} \quad D_{\boldsymbol{y}_c} \boldsymbol{h}(\boldsymbol{0}) = 0 \tag{4.268}$$

Fifth, we determine the $(n-2)$-dimensional function $\boldsymbol{h}$ by constraining the center manifold to be two-dimensional in the n-dimensional space. Substituting for $\boldsymbol{y}_s$ into (4.266) yields

$$\dot{\boldsymbol{y}}_c = J_c \boldsymbol{y}_c + \mu B_{1c} \boldsymbol{y}_c + \mu B_{1s} \boldsymbol{h}(\boldsymbol{y}_c) + \boldsymbol{G}_2 \left[\boldsymbol{y}_c, \boldsymbol{h}(\boldsymbol{y}_c)\right] + \boldsymbol{G}_3 \left[\boldsymbol{y}_c, \boldsymbol{h}(\boldsymbol{y}_c)\right] \tag{4.269}$$

Substituting for $\boldsymbol{y}_s$ into (4.267) and using (4.268), we have

$$\begin{aligned} &D_{\boldsymbol{y}_c} \boldsymbol{h}(\boldsymbol{y}_c) \left\{ J_c \boldsymbol{y}_c + \boldsymbol{G}_2 \left[\boldsymbol{y}_c, \boldsymbol{h}(\boldsymbol{y}_c)\right] + \boldsymbol{G}_3 \left[\boldsymbol{y}_c, \boldsymbol{h}(\boldsymbol{y}_c)\right] \right\} \\ &= J_s \boldsymbol{h}(\boldsymbol{y}_c) + \boldsymbol{F}_2[\boldsymbol{y}_c, \boldsymbol{h}(\boldsymbol{y}_c)] + \boldsymbol{F}_3[\boldsymbol{y}_c, \boldsymbol{h}(\boldsymbol{y}_c)] + \cdots \end{aligned} \tag{4.270}$$

To solve (4.270), one approximates the components of $\boldsymbol{h}(\boldsymbol{y}_c)$ with polynomials. The polynomial approximations are usually taken to be quadratic to the first approximation and do not contain constant and linear terms so that the conditions on $\boldsymbol{h}$ are satisfied. Substituting the assumed quadratic polynomial approximations into (4.270) and equating the coefficients of the different powers in the polynomials on both sides, one obtains a system of algebraic equations for the coefficients of the polynomials. Solving these equations, we obtain a first approximation to the center manifold $\boldsymbol{y}_s = \boldsymbol{h}(\boldsymbol{y}_c)$. Finally, substituting this approximation into (4.269), we obtain a two-dimensional dynamical system describing the dynamics on the center manifold.

Example 4.18

We use a combination of center-manifold reduction and the method of normal forms to compute the normal form of the Hopf bifurcation near the origin of (4.240)–(4.242). In this case, the local manifold is one-dimensional and tangent to the $x_1 - x_2$ plane at the origin. We approximate this manifold by a quadratic function in the form

$$x_3(x_1, x_2) = h(x_1, x_2) = a_1 x_1^2 + a_2 x_2^2 + a_3 x_1 x_2 + \cdots \tag{4.271}$$

Substituting (4.271) into (4.242) yields

$$\begin{aligned} 2a_1 x_1 \dot{x}_1 + 2a_2 x_2 \dot{x}_2 + a_3 x_1 \dot{x}_2 + a_3 x_2 \dot{x}_1 = &-\left(a_1 x_1^2 + a_2 x_2^2 + a_3 x_1 x_2\right) \\ &+ x_1^2 + 2x_2^2 + \cdots \end{aligned} \tag{4.272}$$

Substituting (4.240) and (4.241) into (4.272) and using (4.271), we have

$$2(a_2 - a_1)x_1 x_2 + a_3 x_1^2 - a_3 x_2^2 = -\left(a_1 x_1^2 + a_2 x_2^2 + a_3 x_1 x_2\right) + x_1^2 + 2x_2^2 + \cdots \tag{4.273}$$

Equating the coefficients of each of x_1^2, x_2^2, and $x_1 x_2$ on both sides of (4.273), we obtain

$$a_3 + a_1 = 1\,, \quad a_2 - a_3 = 2\,, \quad a_3 + 2a_2 - 2a_1 = 0$$

whose solution is

$$a_1 = \tfrac{7}{5}\,, \quad a_2 = \tfrac{8}{5}\,, \quad a_3 = -\tfrac{2}{5} \tag{4.274}$$

Substituting (4.274) into (4.271) and then substituting the result into (4.240) and (4.241), we obtain the following two-dimensional system describing the dynamics of (4.240)–(4.242) on the center manifold:

$$\dot{x}_1 = \mu x_1 - x_2 + x_1^2 - \tfrac{7}{5} x_1^3 \alpha_1 + \tfrac{2}{5} x_1^2 x_2 \alpha_1 - \tfrac{8}{5} x_1 x_2^2 \alpha_1 \tag{4.275}$$

$$\dot{x}_2 = \mu x_2 + x_1 + \tfrac{7}{5} x_1^2 x_2 \alpha_2 - \tfrac{2}{5} x_1 x_2^2 \alpha_2 + \tfrac{8}{5} x_2^3 \alpha_2 \tag{4.276}$$

Next, we use the method of normal forms to simplify (4.275) and (4.276). First, we calculate the Jacobian matrix of (4.275) and (4.276) evaluated at the origin. The result is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Its eigenvalues are $\lambda = \pm i$. The right and left eigenvectors of this matrix corresponding to $\lambda = i$ are

$$\boldsymbol{p} = \begin{pmatrix} i \\ 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{q} = \tfrac{1}{2}\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

Second, we introduce the transformation

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \boldsymbol{p}\,u(t) + \bar{\boldsymbol{p}}\,\bar{u}(t)$$

and obtain

$$x_1(t) = i\,u(t) - i\,\bar{u}(t) \quad \text{and} \quad x_2(t) = u(t) + \bar{u}(t) \tag{4.277}$$

Substituting (4.277) into (4.275) and (4.276) and multiplying the result from the left with $\boldsymbol{q}$ yields

$$\begin{aligned} \dot{u} = {} & i u + \mu u + \tfrac{1}{2} i(u^2 - 2u\bar{u} + \bar{u}^2) + \tfrac{1}{10}\left[(31 - 2i)\alpha_2 - (29 + 2i)\alpha_1\right] u^2 \bar{u} \\ & + \text{nonresonance cubic and higher-order terms} \end{aligned} \tag{4.278}$$

Third, we introduce a near-identity transformation of the form

$$u(t) = v(t) + b_1 v^2(t) + b_2 v(t)\bar{v}(t) + b_3 \bar{v}^2 \tag{4.279}$$

into (4.278), approximate $\dot{v}$ and $\dot{\bar{v}}$ with iv and $-i\bar{v}$, respectively, and obtain

$$\begin{aligned} \dot{v} = {} & i v + \mu v + \tfrac{1}{2} i(1 - 2b_1)v^2 - i(1 - b_2)v\bar{v} + \tfrac{1}{2} i(1 + 6b_3)\bar{v}^2 \\ & + \tfrac{1}{10}\left[(31 - 2i)\alpha_2 - (29 + 2i)\alpha_1 + 10i(b_3 - b_1)\right] v^2 \bar{v} \end{aligned} \tag{4.280}$$

Fourth, we choose the b_i to eliminate the quadratic terms and obtain

$$b_1 = \tfrac{1}{2}\,, \quad b_2 = 1\,, \quad b_3 = -\tfrac{1}{6} \tag{4.281}$$

Finally, we substitute for the b_i in (4.280) and obtain the normal form

$$\dot{v} = i v + \mu v + \tfrac{1}{30}\left[93\alpha_2 - 87\alpha_1 - (20 + 6\alpha_1 + 6\alpha_2)i\right] v^2 \bar{v} \tag{4.282}$$

which is in full agreement with (4.261) obtained with the method of multiple scales.

4.6.3 Projection Method

To determine the normal form of the Hopf bifurcation at the origin of the system (4.226) as μ increases past zero with the projection method, we assume that

$$\boldsymbol{x}(t) = \boldsymbol{p}\,u(t) + \bar{\boldsymbol{p}}\,\bar{u}(t) + \boldsymbol{y}(t) \tag{4.283}$$

where $\boldsymbol{p}$ and $\bar{\boldsymbol{p}}$ are the right eigenvectors of A corresponding to the eigenvalues $\pm i\omega$. We assume that all of the other eigenvalues of A are in the left-half of the complex plane. Moreover, we constrain $\boldsymbol{\gamma}(t)$ to be orthogonal to the left eigenvectors $\boldsymbol{q}$ and $\bar{\boldsymbol{q}}$ corresponding to the eigenvalues $\pm i\omega$ normalized so that $\boldsymbol{q}^{\mathrm{T}}\boldsymbol{p} = 1$ and $\bar{\boldsymbol{q}}^{\mathrm{T}}\bar{\boldsymbol{p}} = 1$; that is, we require

$$\boldsymbol{q}^{\mathrm{T}}\boldsymbol{\gamma} = 0 \quad \text{and} \quad \bar{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{\gamma} = 0 \tag{4.284}$$

Substituting (4.283) into (4.226) yields

$$\begin{aligned} \boldsymbol{p}\dot{u} + \bar{\boldsymbol{p}}\dot{\bar{u}} + \dot{\boldsymbol{\gamma}} = {} & i\omega\boldsymbol{p}u - i\omega\bar{\boldsymbol{p}}\bar{u} + A\boldsymbol{\gamma} + \mu uB\boldsymbol{p} + \boldsymbol{Q}(\boldsymbol{p},\boldsymbol{p})u^2 \\ & + 2\boldsymbol{Q}(\boldsymbol{p},\bar{\boldsymbol{p}})u\bar{u} + \boldsymbol{Q}(\bar{\boldsymbol{p}},\bar{\boldsymbol{p}})\bar{u}^2 + 3\boldsymbol{C}(\boldsymbol{p},\boldsymbol{p},\bar{\boldsymbol{p}})u^2\bar{u} \\ & + 2\boldsymbol{Q}(\boldsymbol{p},\boldsymbol{\gamma})u + \boldsymbol{Q}(\bar{\boldsymbol{p}},\boldsymbol{\gamma})\bar{u} + \cdots \end{aligned} \tag{4.285}$$

where the three dots stand for nonresonance and higher-order terms. Multiplying (4.285) from the left with $\boldsymbol{q}^{\mathrm{T}}$, we have

$$\begin{aligned} \dot{u} = {} & i\omega\boldsymbol{p}u + \mu uB\boldsymbol{p} + \boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{p},\boldsymbol{p})u^2 + 2\boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{p},\bar{\boldsymbol{p}})u\bar{u} + \boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}(\bar{\boldsymbol{p}},\bar{\boldsymbol{p}})\bar{u}^2 \\ & + 3\boldsymbol{q}^{\mathrm{T}}\boldsymbol{C}(\boldsymbol{p},\boldsymbol{p},\bar{\boldsymbol{p}})u^2\bar{u} + 2\boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{p},\boldsymbol{\gamma})u + \boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}(\bar{\boldsymbol{p}},\boldsymbol{\gamma})\bar{u} + \cdots \end{aligned} \tag{4.286}$$

Substituting (4.286) and its complex conjugate into (4.285), we obtain

$$\begin{aligned} \dot{\boldsymbol{\gamma}} = {} & A\boldsymbol{\gamma} + \left[\boldsymbol{Q}(\boldsymbol{p},\boldsymbol{p}) - \boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{p},\boldsymbol{p})\boldsymbol{p} - \bar{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{p},\boldsymbol{p})\bar{\boldsymbol{p}}\right]u^2 \\ & + 2\left[\boldsymbol{Q}(\boldsymbol{p},\bar{\boldsymbol{p}}) - \boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{p},\bar{\boldsymbol{p}})\boldsymbol{p} - \bar{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{p},\bar{\boldsymbol{p}})\bar{\boldsymbol{p}}\right]u\bar{u} \\ & + \left[\boldsymbol{Q}(\bar{\boldsymbol{p}},\bar{\boldsymbol{p}}) - \boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}(\bar{\boldsymbol{p}},\bar{\boldsymbol{p}})\boldsymbol{p} - \bar{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{Q}(\bar{\boldsymbol{p}},\bar{\boldsymbol{p}})\bar{\boldsymbol{p}}\right]\bar{u}^2 + \cdots \end{aligned} \tag{4.287}$$

A particular solution of (4.287) can be expressed as

$$\begin{aligned} \boldsymbol{\gamma} = {} & + \left[\eta_2 + \frac{i}{\omega}\boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{p},\boldsymbol{p})\boldsymbol{p} + \frac{i}{3\omega}\bar{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{p},\boldsymbol{p})\bar{\boldsymbol{p}}\right]u^2 \\ & + 2\left[\eta_0 - \frac{i}{\omega}\boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{p},\bar{\boldsymbol{p}})\boldsymbol{p} + \frac{i}{\omega}\bar{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{p},\bar{\boldsymbol{p}})\bar{\boldsymbol{p}}\right]u\bar{u} \\ & + \left[\bar{\eta}_2 - \frac{i}{3\omega}\boldsymbol{q}^{\mathrm{T}}\boldsymbol{Q}(\bar{\boldsymbol{p}},\bar{\boldsymbol{p}})\boldsymbol{p} - \frac{i}{\omega}\bar{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{Q}(\bar{\boldsymbol{p}},\bar{\boldsymbol{p}})\bar{\boldsymbol{p}}\right]\bar{u}^2 + \cdots \end{aligned} \tag{4.288}$$

Next, we substitute (4.288) into (4.284) and determine constraint conditions on η_2 and η_0. Finally, we substitute (4.288) into (4.286) and introduce a near-identity transformation to eliminate the quadratic terms, thereby obtaining the normal form of the bifurcation.

We note that, out of the three methods, the method of multiple scales seems to be the best for simplifying high-dimensional nonlinear systems near a Hopf bifurcation.

Example 4.19

We use the projection method to compute the normal form of the Hopf bifurcation at the origin of the system (4.240)–(4.242) as μ passes through zero. In this case,

$$\boldsymbol{p} = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{q} = \frac{1}{2}\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} \tag{4.289}$$

and (4.283) and (4.284) lead to

$$\begin{aligned} x_1(t) &= i u(t) - i\bar{u}(t) + y_1(t)\,, \\ x_2(t) &= u(t) + \bar{u}(t) + y_2(t)\,, \quad \text{and} \\ x_3(t) &= y_3(t) \end{aligned} \tag{4.290}$$

$$y_1(t) = 0 \quad \text{and} \quad y_2(t) = 0 \tag{4.291}$$

Substituting (4.290) and (4.291) into (4.240)–(4.242) yields

$$i\dot{u} - i\dot{\bar{u}} = -u - \bar{u} + i\mu(u - \bar{u}) - u^2 + 2u\bar{u} - \bar{u}^2 - i\alpha_1 u y_3 + i\alpha_1 \bar{u} y_3 \tag{4.292}$$

$$\dot{u} + \dot{\bar{u}} = -i\bar{u} + \mu(u\bar{u}) + \alpha_2 u y_3 + \alpha_2 \bar{u} y_3 \tag{4.293}$$

$$\dot{y}_3 = -y_3 + u^2 + 6u\bar{u} + \bar{u}^2 \tag{4.294}$$

Multiplying (4.292)–(4.294) from the left with $\boldsymbol{q}$, we have

$$\begin{aligned} \dot{u} = {} & iu + \mu u + \tfrac{1}{2} i u^2 - iu\bar{u} + \tfrac{1}{2} i u^2 - \tfrac{1}{2} u y_3 \alpha_1 + \tfrac{1}{2}\bar{u} y_3 \alpha_1 \\ & + \tfrac{1}{2} u y_3 \alpha_2 + \tfrac{1}{2}\bar{u} y_3 \alpha_2 \end{aligned} \tag{4.295}$$

Substituting (4.295) and its complex conjugate into (4.292)–(4.294), we find that the first two equations reduce to zero. Then, solving (4.294) yields

$$y_3 = \tfrac{1}{5}(1 - 2i)u^2 + 6u\bar{u} + \tfrac{1}{5}(1 + 2i)\bar{u}^2 \tag{4.296}$$

Substituting (4.296) into (4.295), we obtain (4.282) obtained with center-manifold reduction.

4.7 Exercises

4.7.1 Consider the following one-dimensional systems:

a) $\dot{x} = \mu x + x^2$
b) $\dot{x} = -\mu + x^2$

c) $\dot{x} = -\mu x + x^3$
d) $\dot{x} = -\mu x - x^3$
e) $\dot{x} = \mu - x^3$.

In each case, x is the state variable and μ is the control parameter. Construct the bifurcation diagrams for all cases and discuss them.

4.7.2 Consider the following one-dimensional systems:

a) $\dot{x} = 2\mu^2 + \mu x - x^2$
b) $\dot{x} = \dot{x} = 2\mu^2 + \mu x - x^3$
c) $\dot{x} = \mu + \mu x - x^3$
d) $\dot{x} = \mu + \mu x - x^2 + x^3$.

What type of bifurcation does each of these systems undergo with increasing μ at $\mu = 0$? Determine the bifurcating solutions.

4.7.3 Determine the fixed points of

$$\dot{x} = x^2 - 4x + 3$$

Determine their stability.

4.7.4 Find the equilibrium solutions of

$$\dot{x} = x - 2ax^2 + x^3$$

Determine their stability and the bifurcations, which they undergo as a is varied.

4.7.5 Sketch the bifurcation diagram of

$$\dot{x} = x^3 + a^3 - 3ax$$

in the (a, x) plane, indicating which equilibrium solutions are stable and identifying a turning point and a pitchfork bifurcation.

4.7.6 Sketch the bifurcation diagram of

$$\dot{x} = 2(a^2 - x^2) - (a^2 + x^2)^2$$

in the (a, x) plane, indicating which equilibrium solutions are stable and identifying two turning points and a transcritical bifurcation.

4.7.7 Sketch the bifurcation diagram of

$$\dot{x} = -x(x^2 - 2bx - a)$$

in the (a, x) plane for a given positive b, indicating which solutions are stable.

4.7.8 Consider the one-dimensional system

$$\dot{x} = ax + bx^3 + cx^5$$

Determine the fixed points and their stability.

4.7.9 Consider the system

$$\dot{x} = x^4 + ax^2 + bx + c$$

Sketch the intersection of the bifurcation set and three planes a = constant for $a < 0$, $a = 0$, and $a > 0$ (Thom, 1975).

4.7.10 Consider the one-dimensional system (Drazin, 1992)

$$\dot{x} = x^3 - 2ax^2 - (b-3)x + c$$

for real a, b, and c.

a) Show that, if $c = 0$, then there is a transcritical bifurcation, but if $c \neq 0$, there are two (nonbifurcating) branches of equilibria.
b) Show that the loci of the bifurcation points is given by the curve

$$(27c - 18b + 38)^2 = 4(3b-5)^3$$

which has a cusp at $b = 3/5$ and $c = -8/27$.

4.7.11 Determine the fixed points and their types for the following systems, and for each case sketch the trajectories and the separatrices in the phase plane:

a) $\ddot{x} + 2\dot{x} + x + x^3 = 0$
b) $\ddot{x} + 2\dot{x} + x - x^3 = 0$
c) $\ddot{x} + 2\dot{x} - x + x^3 = 0$
d) $\ddot{x} + 2\dot{x} - x - x^3 = 0$
e) $\ddot{x} - a + x^2 = 0$ for $a > 0$, $a = 0$, and $a < 0$.

4.7.12 Consider the following system (Drazin, 1992):

$$\dot{x} = x^3 + \delta x^2 - \mu x$$

Determine the fixed points of this system and study the bifurcations in the (x, μ) plane for zero and nonzero values of δ. Show that the pitchfork bifurcation at (0, 0) for $\delta = 0$ becomes a transcritical bifurcation for small δ and that there is a turning point at $(-1/2\delta, -1/4\delta^2)$. Sketch the bifurcation diagram in the (x, μ) plane for $\delta > 0$.

4.7.13 Consider the following single-degree-of-freedom system with quadratic and cubic nonlinearities:

$$\ddot{x} + x + \delta x^2 + \alpha x^3 = 0$$

Sketch the potential energy $V(x)$ for the system and the associated phase portrait for each of the following cases: (a) $\delta = 3$ and $\alpha = 4$, (b) $\delta = \alpha = 4$, and (c) $\delta = 5$ and $\alpha = 4$.

It is common to refer to the first case as a *single-well potential system* because there is a *well* in the graph of $V(x)$ versus x. The third case is referred to as a *two-well potential system*. From the phase portraits, one can discern a qualitative change as one goes from the first case to the third case.

4.7.14 Consider the system

$$\dot{x} = x(3 - x - 2y)$$
$$\dot{y} = y(2 - x - y)$$

where $x > 0$ and $y > 0$. Determine all of the fixed points of this system and determine their types.

4.7.15 Consider the system

$$\dot{x} = x^2 - \mu^2$$
$$\dot{y} = y + x^2 - \mu^2$$

Determine the equilibrium solutions and their stability.

4.7.16 Consider the system

$$\dot{x}_1 = x_2 - \tfrac{1}{3}\lambda(x_1^3 - 3x_1)$$
$$\dot{x}_2 = -x_1$$

Show that the origin is the only equilibrium point. Determine its stability as a function of λ.

4.7.17 Show that the origin is an unstable equilibrium point for each of the two systems

a) $\dot{x}_1 = x_2, \dot{x}_2 = x_1 + 2x_2^3$.

b) $\dot{x}_1 = x_1 + 5x_2 + x_1^2 x_2, \dot{x}_2 = 5x_1 + x_2 - x_2^3$.

4.7.18 Show that the origin is a saddle point for each of the two systems

a) $\dot{x}_1 = -x_2, \dot{x}_2 = x_2 + x_1^3$.

b) $\dot{x}_1 = x_2, \dot{x}_2 = x_1 + x_1^3$.

Determine the stable and unstable manifolds of the origin for the linearized as well as the nonlinear systems.

4.7.19 Determine an approximation to the stable and unstable manifolds of the saddles of the system

$$\dot{x}_1 = 1 - x_1 x_2$$
$$\dot{x}_2 = x_1 - x_2^3$$

4.7.20 Consider the system

$$\dot{x}_1 = \lambda + 2x_1 x_2$$
$$\dot{x}_2 = 1 + x_1^2 - x_2^2$$

Show that there are two saddles when $\lambda = 0$ and that the x_2-axis is invariant. Hence, show that there is a heteroclinic connection. Sketch the phase plane. Show that, when $|\lambda| \neq 0$ and is small, there are still two saddle points but the saddle connection is no longer present. Sketch the phase plane for $1 \gg \lambda > 0$ and $-1 \ll \lambda < 0$.

4.7.21 Show that the trivial solution is the only equilibrium solution of

$$\dot{x} = xy^2 + x^2y + x^3 \quad \text{and} \quad \dot{y} = y^3 - x^3$$

and that it is unstable.

4.7.22 Show that the trivial solution of the system

$$\dot{x} = 2xy^2 - x^3 \ \text{and} \ \dot{y} = \frac{2}{5}x^2y - y^3$$

is asymptotically stable.

4.7.23 Show that the origin is a stable equilibrium point of the system

$$\dot{x} = y - x^3 \quad \text{and} \quad \dot{y} = -x^2$$

4.7.24 Show that the origin is an asymptotically stable equilibrium point of the system

$$\dot{x} = y - x(x^4 + y^4) \quad \text{and} \quad \dot{y} = -x - y(x^4 + y^4)$$

4.7.25 The origin of each of the following systems is a *degenerate saddle point:*

a) $\dot{x} = x^2, \dot{y} = -y$
b) $\dot{x} = x^2 - y^2, \dot{y} = 2xy$.

Sketch the phase portrait for each case.

4.7.26 Show that the origin is an unstable equilibrium point of the system

$$\dot{x} = 2x^2y \quad \text{and} \quad \dot{y} = -2xy^2$$

Hint: Show that xy is constant on each orbit.

4.7.27 Consider the system

$$\dot{x} = xy \quad \text{and} \quad \dot{y} = 2 - x - y$$

Find the fixed points and determine their stability.

4.7.28 Consider the planar system

$$\dot{x} = \tfrac{1}{2}(-x + x^3)$$
$$\dot{y} = \frac{2y}{1 - 2x^2} + \epsilon$$

where $\epsilon \ll 1$. Show that there are three saddle points. For $\epsilon = 0$ and $\epsilon \neq 0$, sketch the phase portrait and indicate any heteroclinic connections.

4.7.29 Consider the system

$$\ddot{x} + \omega^2 x + \epsilon \delta x^2 \cos(\Omega t) = 0$$

where ϵ is a small nondimensional parameter. Use the method of multiple scales to determine the modulation equations when $\Omega \approx 3\omega$.

4.7.30 Consider the system

$$\ddot{x} + \omega^2 x + \epsilon \delta x^2 \cos(\Omega t) = 0$$

where ϵ is a small nondimensional parameter. Use the method of averaging to determine the modulation equations when $\Omega \approx 3\omega$.

4.7.31 Consider the following speed-control system investigated by Fallside and Patel (1965):

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = K_d x_2 - x_1 - G x_1^2 \left(-\frac{x_2}{K_d} + x_1 + 1 \right)$$

a) For $K_d = -1$ and $G = 6$, determine the fixed points and their stability.
b) Plot the stable manifolds of the unstable fixed points and a few other trajectories in the (x_1, x_2) space.
c) Discuss the phase portrait.

4.7.32 Calculate the fixed points of the system

$$\dot{x} = \alpha x y \quad \text{and} \quad \dot{y} = -y + x^2$$

Determine their stability.

4.7.33 The free oscillation about the upright position of an inverted pendulum constrained to oscillate between two closely spaced rigid barriers is described by (e.g., Shaw and Rand, 1989)

$$\ddot{x} + 2\mu \dot{x} - x = 0 \quad |x| < 1$$
$$\dot{x} \to -r\dot{x} \quad |x| = 1$$

where x describes the position of the pendulum; the locations of the rigid barriers are $x = -1$ and $x = 1$; μ is a measure of the friction; and $r \leq 1$ is a reflection coefficient representing energy loss during impact with either of the rigid barriers.

Assuming elastic impact (i.e., $r = 1$), construct and discuss the phase portraits for the following two cases: (a) $\mu = 0$ and (b) $\mu > 0$.

4.7.34 In studying the forced response of a van der Pol oscillator with delayed amplitude limiting, Nayfeh (1968) encountered the following system of equations:

$$\dot{x}_1 = x_1(1 - x_1^2) + F \cos x_2$$

$$\dot{x}_2 = \sigma + \nu x_1^2 - \frac{F}{x_1} \sin x_2$$

a) Show that the fixed points (x_{10}, x_{20}) of this system satisfy

$$\rho \left[(1-\rho)^2 + (\sigma + \nu\rho)^2 \right] = F^2$$

where $\rho = x_{10}^2$.

b) For $\nu = -0.15$, plot the loci of the fixed points in the $\rho - \sigma$ plane for $F^2 = 1, 1/3, 4/27$, and $1/10$. What is the significance of the value $4/27$?

c) Show that the interior points of the ellipse defined by

$$(1-\rho)(1-3\rho) + (\sigma + \nu\rho)(\sigma + 3\nu\rho) = 0$$

are saddle points and hence unstable. Also, show that the exterior points are nodes if $D \geq 0$ and foci if $D < 0$, where

$$D = 4\left[(1 - 3\nu^2)\rho^2 - 4\nu\rho\sigma - \sigma^2 \right].$$

d) Finally, show that the exterior points are stable if $\rho > 1/2$ and unstable if $\rho < 1/2$.

4.7.35 A bead of mass m sliding on a rotating circular hoop of radius R is described by

$$\ddot{\theta} + 2\mu\dot{\theta} + \frac{g}{R}\sin\theta - \omega^2 \sin\theta \cos\theta = 0$$

Here, θ describes the angular position of the bead on the hoop, g is the acceleration due to gravity, μ is a measure of the friction experienced by the bead, and ω is the angular velocity of the hoop.

a) For $\mu = 0$, determine the fixed points (equilibrium positions) of the system and sketch the phase portrait in each of the following cases: (i) $\omega^2 < g/R$, (ii) $\omega^2 = g/R$, and (iii) $\omega^2 > g/R$.

b) For $\mu > 0$, choose ω as a control parameter and examine the different local bifurcations of the fixed points that occur as ω is increased from zero. Construct appropriate bifurcation diagrams.

4.7.36 Mingori and Harrison (1974) studied the following system for analyzing the motion of a particle constrained to move on a circular path that is spinning and coning:

$$\dot{u} = v$$
$$\dot{v} = -\mu_1(v-1) + \mu_2\mu_3 \sin u + \tfrac{1}{2}\mu_3^2 \sin 2u$$

Let $\mu_1 = 0.1$ and $\mu_2 = 2.0$. Then, as μ_3 is varied from zero, bifurcations take place at 0.0502, 0.3, and 2.265. Examine the qualitative changes that take place in the (u, v) space due to these bifurcations.

4.7.37 Consider the nonlinear oscillator

$$\ddot{u} + u + \epsilon\left[2\mu_1\dot{u} + \mu_2\dot{u}|\dot{u}| + \alpha u^3 + 2Ku\cos(\Omega t)\right] = 0$$

where ϵ is a small, positive parameter. Further, the parameters μ_1, μ_2, and K are all independent of ϵ while the parameter Ω is such that

$$\Omega = 2 + \epsilon\sigma$$

A first approximation obtained for this system has the form

$$u = p\cos\left(\tfrac{1}{2}\Omega t\right) + q\sin\left(\tfrac{1}{2}\Omega t\right) + O(\epsilon)$$

where

$$p' = -\mu_1 p - \frac{1}{2}(\sigma + K)q + \frac{3\alpha}{8}q(p^2+q^2) - \frac{4\mu_2}{3\pi}p\sqrt{p^2+q^2}$$
$$q' = -\mu_1 q + \frac{1}{2}(\sigma - K)p - \frac{3\alpha}{8}p(p^2+q^2) - \frac{4\mu_2}{3\pi}q\sqrt{p^2+q^2}$$

In the above equations, the prime denotes the derivative with respect to the time scale $\tau = \epsilon t$.

a) Simplify the dynamical system governing p and q to its normal form for a transcritical bifurcation in the vicinity of

$$(p, q, K_c) = \left(0, 0, \sqrt{4\mu_1^2 + \sigma^2}\right).$$

b) What happens to the bifurcation at the above-mentioned bifurcation point when $\mu_2 = 0$?
c) Construct the frequency-response curves when $\mu_2 = 0$ and discuss them.

4.7.38 Consider the system

$$\dot{x} = -y + \frac{x}{\sqrt{x^2+y^2}}(1 - x^2 - y^2)$$
$$\dot{y} = x + \frac{y}{\sqrt{x^2+y^2}}(1 - x^2 - y^2)$$

Determine its limit cycle and indicate whether it is stable or unstable.

4.7.39 Consider the system

$$\dot{x} = \mu - x + y + xy$$
$$\dot{y} = 2\mu + x - y - y^2$$

where μ is the control parameter. Determine the normal form of this system in the vicinity of the bifurcation point $(x, y, \mu) = (0, 0, 0)$. What is the type of this bifurcation?

4.7.40 Consider the system

$$\dot{x}_1 = \mu + x_1 - x_2 + 2\mu x_2 + x_1^2$$
$$\dot{x}_2 = b\mu + 2x_1 - 2x_2 + \mu x_1 + \alpha x_2^2$$

where μ is the control parameter. Determine the normal form of this system in the vicinity of the bifurcation point $(x_1, x_2, \mu) = (0, 0, 0)$.

a) Show that this system undergoes a generic saddle-node bifurcation as μ increases past zero when $b \neq 2$ and $\alpha \neq 2$.
b) Is there a bifurcation at $(x_1, x_2, \mu) = (0, 0, 0)$ when $b \neq 2$ and $\alpha = 2$?
c) Show that this system undergoes a generic transcritical bifurcation as μ increases past zero when $b = 2$ and $\alpha \neq 2$.
d) Show that this system undergoes a generic pitchfork bifurcation as μ increases past zero when $b = 2$ and $\alpha = 2$.

4.7.41 Consider the system

$$\dot{x}_1 = x_1 - 2x_2 + \mu b + \alpha_1(x_1 + 2x_2)^2$$
$$\dot{x}_2 = 2x_1 - 4x_2 + \mu b + \alpha_2(x_1 - x_2)^2$$

Determine the type and normal form of the bifurcation that takes place at

$$(x_1, x_2, \mu) = (0, 0, 0)$$

4.7.42 Consider the system

$$\dot{x}_1 = x_1 - 2x_2 + \mu b + \alpha(x_1 + 2x_2)^2$$
$$\dot{x}_2 = 2x_1 - 4x_2 + 2\mu b + 29\alpha(x_1 - x_2)^2$$

Determine the type and normal form of the bifurcation that takes place at

$$(x_1, x_2, \mu) = (0, 0, 0)$$

4.7.43 Consider the system

$$\dot{x}_1 = x_1 - 2x_2 + \mu b + \mu x_1 + \alpha(x_1 + 2x_2)^2$$
$$\dot{x}_2 = 2x_1 - 4x_2 + 2\mu b + \mu x_2 + 32\alpha(x_1 - x_2)^2$$

Determine the type and normal form of the bifurcation that takes place at

$$(x_1, x_2, \mu) = (0, 0, 0)$$

4.7.44 Consider the system

$$\dot{x} = -y + \mu x + \alpha x(x^2 + y^2)$$
$$\dot{y} = x + \mu y + \alpha y(x^2 + y^2)$$

What type of bifurcation occurs at $(x, y, \mu) = (0, 0, 0)$. Determine the bifurcating solutions.

4.7.45 Consider the system

$$\dot{x}_1 = \mu + x_1 - x_2 + 2\mu x_2 + x_1^2$$
$$\dot{x}_2 = 2\mu + 2x_1 - 2x_2 + \mu x_1 + x_2^2$$

where μ is the control parameter. Determine the normal form of this system in the vicinity of the bifurcation point $(x_1, x_2, \mu) = (0, 0, 0)$. What is the type of this bifurcation?

4.7.46 Consider the system

$$\dot{x}_1 = \mu + x_1 - x_2 + 2\mu x_2 + x_1^2$$
$$\dot{x}_2 = \mu + 2x_1 - 2x_2 + \mu x_1 + x_2^2$$

where μ is the control parameter. Determine the normal form of this system in the vicinity of the bifurcation point $(x_1, x_2, \mu) = (0, 0, 0)$. What is the type of this bifurcation?

4.7.47 Consider the system

$$\dot{x}_1 = \mu + x_1 - x_2 + 2\mu x_2 + x_1^2$$
$$\dot{x}_2 = \mu + 2x_1 - 2x_2 + \mu x_1 + 2x_2^2$$

where μ is the control parameter. Determine the normal form of this system in the vicinity of the bifurcation point $(x_1, x_2, \mu) = (0, 0, 0)$. What is the type of this bifurcation?

4.7.48 Consider the system

$$\dot{x}_1 = \mu + x_1 - x_2 + 2\mu x_2 + x_1^2$$
$$\dot{x}_2 = b\mu + 2x_1 - 2x_2 + \mu x_1 + \alpha x_2^2$$

where μ is the control parameter. Determine the normal form of this system in the vicinity of the bifurcation point $(x_1, x_2, \mu) = (0, 0, 0)$ for each of the following cases:

a) $b \neq 2$ and $\alpha \neq 2$
b) $b \neq 2$ and $\alpha = 2$
c) $b = 2$ and $\alpha \neq 2$
d) $b = 2$ and $\alpha = 2$.

4.7.49 Consider the system

$$\begin{aligned}\dot{x}_1 &= x_1 - 2x_2 + \mu b + \alpha(x_1 + 2x_2)^2 \\ \dot{x}_2 &= 2x_1 - 4x_2 + 2\mu b + 29\alpha(x_1 - x_2)^2\end{aligned}$$

Determine the type and normal form of the bifurcation that takes place at

$$(x_1, x_2, \mu) = (0, 0, 0)$$

4.7.50 Consider the system

$$\begin{aligned}\dot{x}_1 &= x_1 - 2x_2 + \mu b + \mu x_1 + \alpha(x_1 + 2x_2)^2 \\ \dot{x}_2 &= 2x_1 - 4x_2 + 2\mu b + \mu x_2 + 32\alpha(x_1 - x_2)^2\end{aligned}$$

Determine the type and normal form of the bifurcation that takes place at

$$(x_1, x_2, \mu) = (0, 0, 0)$$

4.7.51 Consider the system

$$\begin{aligned}\dot{x}_1 &= x_1 - 2x_2 + \mu b + \alpha_1(x_1 + 2x_2)^2 \\ \dot{x}_2 &= 2x_1 - 4x_2 + \mu b + \alpha_2(x_1 - x_2)^2\end{aligned}$$

Determine the type and normal form of the bifurcation that takes place at

$$(x_1, x_2, \mu) = (0, 0, 0)$$

4.7.52 Consider the dynamical system

$$\begin{aligned}\dot{x} &= \tfrac{3}{2}\mu + x - 2y + x^2 \\ \dot{y} &= 2x - 4y + xy\end{aligned}$$

Examine the bifurcation that takes place when $(x, y, \mu) = (0, 0, 0)$ and determine the normal form of the system near this bifurcation.

4.7.53 Consider the system

$$\begin{aligned}\dot{x}_1 &= \mu + x_1 - x_2 + 2\mu x_2 + x_1^2 \\ \dot{x}_2 &= 2\mu + 2x_1 - 2x_2 + \mu x_1 + 2x_2^2\end{aligned}$$

where μ is the control parameter. Determine the normal form of this system in the vicinity of the bifurcation point $(x_1, x_2, \mu) = (0, 0, 0)$. What is the type of this bifurcation?

4.7.54 Consider the system

$$\dot{x} = \mu x - y$$
$$\dot{y} = x + \mu y + \alpha x^3$$

Examine the bifurcation that takes place at

$$(x, y, \mu) = (0, 0, 0)$$

Is the bifurcation supercritical or subcritical?

4.7.55 Consider the system

$$\dot{x} = \mu x - y + xy^2$$
$$\dot{y} = \mu y + x + 2x^2 y$$

where μ is the control parameter. Determine the normal form of this system in the vicinity of the bifurcation point $(x, y, \mu) = (0, 0, 0)$. What is the type of this bifurcation?

4.7.56 Consider the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1 + \lambda x_2 - x_1^2$$

Discuss the bifurcations of the fixed points of this system as a function of the control parameter λ. Sketch the state space for $\lambda = 0$, $\lambda > 0$, and $\lambda < 0$.

4.7.57 Consider the system

$$\dot{x} = \mu x - y + xy^2$$
$$\dot{y} = \mu y + x$$

where μ is the control parameter. Determine the normal form of this system in the vicinity of the bifurcation point $(x, y, \mu) = (0, 0, 0)$. What is the type of this bifurcation?

4.7.58 Consider the system

$$\dot{x} = -2y + \mu x + \alpha x(x^2 + y^2)$$
$$\dot{y} = 2x + \mu y + \alpha y(x^2 + y^2)$$

What type of bifurcation occurs at $(x, y, \mu) = (0, 0, 0)$. Determine the bifurcating solutions.

4.7.59 Consider the Rössler equations (Rössler, 1976a):

$$\dot{x} = -(y + z)$$
$$\dot{y} = x + ay$$
$$\dot{z} = b + (x - c)z$$

Assume that the parameters a, b, and c are positive.

a) When a is used as a control parameter, verify that a fixed point of this system experiences a saddle-node bifurcation at

$$(x, y, z, a) = \left(\frac{c}{2}, -\frac{c}{2a}, \frac{c}{2a}, \frac{c^2}{4b}\right).$$

b) Simplify the three-dimensional system to the normal form for a saddle-node bifurcation in the vicinity of the bifurcation point.

4.7.60 Consider the Lorenz equations (Lorenz, 1963):

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = \rho x - y - xz$$
$$\dot{z} = -\beta z + xy$$

Assume that the parameters σ, β, and ρ are positive.

a) Choose ρ as the control parameter and examine the different local bifurcations experienced by the different fixed points. Verify that a Hopf bifurcation of a fixed point occurs at

$$\rho_c = \frac{\sigma(\sigma + \beta + 3)}{\sigma - \beta - 1}.$$

b) Construct the bifurcation diagram for $0 < \rho \le \rho_c$.
c) Simplify the three-dimensional system to its normal form for a pitchfork bifurcation in the vicinity of $(x, y, z, \rho) = (0, 0, 0, 1)$ and obtain

$$\dot{u} = -\frac{\sigma(1 - \rho)}{\sigma + 1} u - \frac{\sigma}{\beta(\sigma + 1)} u^3.$$

4.7.61 Consider the three-dimensional dynamical system

$$\dot{x}_1 = b_1\mu - x_1 + x_2 - 2x_3 + +b_{11}\mu x_1 + \alpha_1 (x_1 + 2x_2 + x_3)^2 \tag{4.297}$$

$$\dot{x}_2 = b_2\mu - 2x_1 - x_2 - x_3 + b_{22}\mu x_2 + \alpha_2 (x_1 - x_2 + x_3)^2 \tag{4.298}$$

$$\dot{x}_3 = b_3\mu + x_1 + 2x_2 - x_3 + b_{33}\mu x_3 + \alpha_3 (2x_1 + x_2 - x_3)^2 \tag{4.299}$$

Show that the origin of this system undergoes a saddle-node bifurcation as μ increases past zero when $(b_3 + b_2 - b_1) \neq 0$. Show that this system can be simplified to

$$\dot{u} = \tfrac{1}{3} (b_3 + b_2 - b_1) \mu_2 + \tfrac{1}{3} (4\alpha_3 + \alpha_2 - 4\alpha_1) u^2 \tag{4.300}$$

Show that the origin of this system undergoes a transcritical bifurcation as μ increases past zero when $(b_3 + b_2 - b_1) = 0$. Show that this system can be simplified to

$$\dot{u} = \Gamma_0 \mu_1^2 + \Gamma_1 \mu_1 u + \Gamma_2 u^2 \tag{4.301}$$

where

$$\begin{aligned}
\Gamma_0 &= \tfrac{1}{27}\big(-3b_2 b_{11} - 3b_3 b_{22} + 3b_2 b_{33} + 3b_3 b_{33} - 4b_2^2 \alpha_1 + 4b_2 b_3 \alpha_1 - b_3^2 \alpha_1 \\
&\quad + 4b_2^2 \alpha_2 + 8b_2 b_3 \alpha_2 + 4b_3^2 \alpha_2 + b_2^2 \alpha_3 - 4b_2 b_3 \alpha_3 + 4b_3^2 \alpha_3\big) \\
\Gamma_1 &= \tfrac{1}{9}(3b_{11} + 3b_{22} + 3b_{33} - 8b_2 \alpha_1 + 4b_3 \alpha_1 \\
&\quad - 4b_2 \alpha_2 - 4b_3 \alpha_2 - 4b_2 \alpha_3 + 8b_3 \alpha_3) \\
\Gamma_2 &= \tfrac{1}{3}(4\alpha_3 + \alpha_2 - 4\alpha_1)
\end{aligned}$$

4.7.62 Consider the system

$$\begin{aligned}
\dot{x} &= \mu x - 2y \\
\dot{y} &= \mu y + 2x + \lambda_1 xz + \lambda_2 yz \\
\dot{z} &= -z + \alpha x^2
\end{aligned}$$

where α, μ, λ_1, and τ are constants. Use a combination of center-manifold reduction and the method of normal forms to construct periodic solutions of this system for small μ.

4.7.63 Consider the system

$$\begin{aligned}
\dot{x}_1 &= -x_1 + h(x_3) \\
\dot{x}_2 &= -h(x_3) \\
\dot{x}_3 &= -a x_1 + b x_2 - c h(x_3)
\end{aligned}$$

where a, b, and c are positive constants and $h(0) = 0$ and

$$y h(y) > 0 \quad \text{for} \quad 0 < |y| < k \quad \text{for some} \quad k > 0$$

a) Show that the origin is an isolated equilibrium point.
b) Is the origin an asymptotically stable equilibrium point?
c) Suppose that $y h(y) > 0$. Is the origin globally asymptotically stable?

4.7.64 Consider the system

$$\begin{aligned}
\dot{x} &= y \\
\dot{y} &= -x + \mu y - \alpha y z \\
\dot{z} &= -z + x^2
\end{aligned}$$

Show that the center manifold of the origin ($x = 0$, $y = 0$, $\mu = 0$) is given by

$$z = \frac{1}{5}\left(3x^2 - 2xy + 2y^2\right)$$

Then, determine the equations governing the dynamics on this manifold.

4.7.65 Consider the system

$$\dot{x} = \mu x - y + x^2 - \alpha_1 xz$$
$$\dot{y} = \mu y + x + \alpha_2 yz$$
$$\dot{z} = -z + x^2 + 2y^2$$

a) Show that the origin of this system undergoes a Hopf bifurcation as μ increases through zero.
b) Determine the normal form of this bifurcation.
c) Is the bifurcation supercritical or subcritical?
d) Calculate the amplitude and frequency of the bifurcating limit cycle.

4.7.66 Consider the system

$$\dot{x}_1 = \mu x_1 - x_2$$
$$\dot{x}_2 = \mu x_2 + x_1 + 2x_1 x_3$$
$$\dot{x}_3 = -x_3 + x_1^2$$

where μ is a small parameter. Design a controller to convert the bifurcation at $(x_1, x_2, x_3, \mu) = (0, 0, 0, 0)$ from subcritical to supercritical.

4.7.67 Consider the system

$$\ddot{u} + \omega^2 u = \mu \dot{u} - \alpha_1 uv - \alpha_2 u\dot{v}$$
$$\dot{v} + v = \delta u^2$$

Determine the normal form of this system near $(u, v) = (0, 0)$ as μ passes through zero.

5
Forced Oscillations of the Duffing Oscillator

In this chapter, we consider the response of a single-degree-of-freedom system to a harmonic excitation modeled by

$$\ddot{u} + \omega^2 u + 2\epsilon\mu\dot{u} + \epsilon\alpha u^3 = F\cos\Omega t$$

where ϵ is a small nondimensional parameter that is used as a bookkeeping device. Carrying out a straightforward expansion, one finds that up to $O(\epsilon)$, resonances occur if

- $\Omega \approx \omega$: Primary or main resonance
- $\Omega \approx 3\omega$: Subharmonic resonance of order one-third
- $\Omega \approx \frac{1}{3}\omega$: Superharmonic resonance of order three.

Next, we use the method of normal forms to determine second-order uniform approximations to the solutions of this equation for these resonances.

5.1
Primary Resonance

In the case of primary resonance, the linear theory shows that a small excitation leads to a large response. Hence, we need to determine the orders of F and u that will explicitly display this observation. One way of accomplishing this is to order the excitation at $O(\epsilon)$ and rewrite the governing equation as

$$\ddot{u} + \omega^2 u = -\epsilon\left[2\mu\dot{u} + \alpha u^3 - F\cos\Omega t\right] \tag{5.1}$$

To apply the method of normal forms, we let

$$z = Fe^{i\Omega t} \tag{5.2}$$

so that

$$F\cos\Omega t = \tfrac{1}{2}(z + \bar{z}) \tag{5.3}$$

The Method of Normal Forms, Second Edition. Ali Hasan Nayfeh

and

$$\dot{z} = i\Omega z \tag{5.4}$$

Hence, we transform the two-dimensional nonautonomous problem into a three-dimensional autonomous problem. Next, to quantitatively describe the nearness of the primary resonance, we introduce a detuning parameter σ defined according to

$$\Omega^2 = \omega^2 + \epsilon\sigma \tag{5.5}$$

Using (5.5) and (5.2) to eliminate ω^2 and $F\cos\Omega t$, respectively, from (5.1), we have

$$\ddot{u} + \Omega^2 u = -\epsilon\left[2\mu\dot{u} - \sigma u + \alpha u^3 - \tfrac{1}{2}(z + \bar{z})\right] \tag{5.6}$$

Equation 5.6 can conveniently be represented as a first-order complex-valued equation. To this end, we note that when, $\epsilon = 0$, the solution of (5.6) can be expressed as

$$u = Ae^{i\Omega t} + \bar{A}e^{-i\Omega t}$$

where A is a constant. Then,

$$\dot{u} = i\Omega\left(Ae^{i\Omega t} - \bar{A}e^{-i\Omega t}\right)$$

When $\epsilon \neq 0$, A will be time-varying rather than constant. To represent (5.6) as a single complex-valued equation, we identify $Ae^{i\Omega t}$ as ζ and hence introduce the transformation

$$u = \zeta + \bar{\zeta} \quad \text{and} \quad \dot{u} = i\Omega\left(\zeta - \bar{\zeta}\right) \tag{5.7}$$

so that

$$\zeta = \frac{1}{2}\left(u - \frac{i}{\Omega}\dot{u}\right) \quad \text{and} \quad \bar{\zeta} = \frac{1}{2}\left(u + \frac{i}{\Omega}\dot{u}\right) \tag{5.8}$$

With this transformation, (5.6) becomes

$$\dot{\zeta} = i\Omega\zeta + \frac{i\epsilon}{2\Omega}\left[2i\Omega\mu\left(\zeta - \bar{\zeta}\right) - \sigma\left(\zeta + \bar{\zeta}\right) + \alpha\left(\zeta + \bar{\zeta}\right)^3 - \frac{1}{2}\left(z + \bar{z}\right)\right] \tag{5.9}$$

We note that the unperturbed problem is transformed into the simple equation $\dot{\zeta} = i\Omega\zeta$ under the transformation (5.7). Other transformations may lead to an equation involving ζ and $\bar{\zeta}$.

Next, we use the near-identity transformation

$$\zeta = \eta + \epsilon h\left(\eta, \bar{\eta}, z, \bar{z}\right) + \cdots \tag{5.10}$$

so that (5.9) takes the simple form

$$\dot{\eta} = i\Omega\eta + \epsilon g(\eta, \bar{\eta}, z, \bar{z}) + \cdots \tag{5.11}$$

Substituting (5.10) and (5.11) into (5.9), using (5.4), and equating the coefficients of ϵ on both sides, we obtain

$$\begin{aligned} &g + i\Omega\left(\frac{\partial h}{\partial \eta}\eta - \frac{\partial h}{\partial \bar{\eta}}\bar{\eta} + \frac{\partial h}{\partial z}z - \frac{\partial h}{\partial \bar{z}}\bar{z} - h\right) \\ &= -\mu(\eta - \bar{\eta}) + \frac{i}{2\Omega}\left[-\sigma(\eta + \bar{\eta}) + \alpha(\eta + \bar{\eta})^3 - \frac{1}{2}(z + \bar{z})\right] \end{aligned} \tag{5.12}$$

Next, we choose h to eliminate all of the nonresonance terms in (5.12), leaving g with the resonance and near-resonance terms. The right-hand side of (5.12) suggests choosing h in the form

$$h = \Lambda_1\eta^3 + \Lambda_2\eta^2\bar{\eta} + \Lambda_3\eta\bar{\eta}^2 + \Lambda_4\bar{\eta}^3 + \Delta_1\eta + \Delta_2\bar{\eta} + \Delta_3 z + \Delta_4\bar{z} \tag{5.13}$$

Substituting (5.13) into (5.12) and rearranging yields

$$\begin{aligned} &g + \mu\eta + \frac{i\sigma}{2\Omega}\eta - \frac{3i\alpha}{2\Omega}\eta^2\bar{\eta} + \frac{i}{4\Omega}z - i\left(-2\Omega\Lambda_1 + \frac{\alpha}{2\Omega}\right)\eta^3 \\ &- i\left(2\Omega\Lambda_3 + \frac{3\alpha}{2\Omega}\right)\eta\bar{\eta}^2 - i\left(4\Omega\Lambda_4 + \frac{\alpha}{2\Omega}\right)\bar{\eta}^3 \\ &- \left(2i\Omega\Delta_2 + \mu - \frac{i\sigma}{2\Omega}\right)\bar{\eta} - i\left(2\Omega\Delta_4 - \frac{1}{4\Omega}\right)\bar{z} = 0 \end{aligned} \tag{5.14}$$

We note that (5.14) does not depend on Δ_1, Δ_3, and Λ_2, and hence they are arbitrary, and the terms proportional to $\eta, \eta^2\bar{\eta}$, and z are resonance terms. Choosing $\Lambda_1, \Lambda_3, \Lambda_4, \Delta_2$, and Δ_4 to eliminate the nonresonance terms, we have

$$\Delta_2 = \frac{i\mu}{2\Omega} + \frac{\sigma}{4\Omega^2}, \quad \Delta_4 = \frac{1}{8\Omega^2} \tag{5.15}$$

$$\Lambda_1 = \frac{\alpha}{4\Omega^2}, \quad \Lambda_3 = -\frac{3\alpha}{4\Omega^2}, \quad \Lambda_4 = -\frac{\alpha}{8\Omega^2} \tag{5.16}$$

With the choices (5.15) and (5.16), (5.14) yields g as

$$g = -\mu\eta - \frac{i\sigma}{2\Omega}\eta + \frac{3i\alpha}{2\Omega}\eta^2\bar{\eta} - \frac{i}{4\Omega}z \tag{5.17}$$

which, upon substitution into (5.11), yields the normal form

$$\dot{\eta} = i\Omega\eta - \epsilon\mu\eta - \frac{i\epsilon\sigma}{2\Omega}\eta + \frac{3i\epsilon\alpha}{2\Omega}\eta^2\bar{\eta} - \frac{i\epsilon}{4\Omega}z \tag{5.18}$$

Substituting (5.13) into (5.10) and then substituting the result into (5.7), we obtain

$$\begin{aligned} u = \eta + \bar{\eta} + \epsilon\big[&(\Lambda_1 + \Lambda_4)(\eta^3 + \bar{\eta}^3) + (\Lambda_2 + \Lambda_3)\eta^2\bar{\eta} + (\bar{\Lambda}_2 + \Lambda_3)\eta\bar{\eta}^2 \\ &+ (\Delta_1 + \bar{\Delta}_2)\eta + (\Delta_2 + \bar{\Delta}_1)\bar{\eta} + (\Delta_3 + \Delta_4)z + (\bar{\Delta}_3 + \Delta_4)\bar{z}\big] + \cdots \end{aligned} \tag{5.19}$$

If we express η in the polar form

$$\eta = \tfrac{1}{2} a e^{i(\Omega t + \gamma)} \tag{5.20}$$

we find that, to $O(1)$, the amplitude of the fundamental frequency Ω of oscillations is a. However, some of the first-order terms in (5.19), namely, $\eta^2\bar{\eta}, \eta\bar{\eta}^2, \eta, \bar{\eta}, z$, and $\bar{z}$, also have the frequency Ω, and hence modify a and make it nonunique. In order to uniquely define the amplitude of the term at the fundamental frequency Ω of oscillations by a, we choose Λ_2, Δ_1, and Δ_3 to eliminate the terms involving $z, \bar{z}, \eta, \bar{\eta}, \eta^2\bar{\eta}$, and $\eta\bar{\eta}^2$ in (5.19). With this choice, (5.19) becomes

$$u = \eta + \bar{\eta} + \frac{\epsilon\alpha}{8\Omega^2}\left(\eta^3 + \bar{\eta}^3\right) + \cdots \tag{5.21}$$

on account of (5.16). In terms of the polar coordinates (5.20), (5.21) becomes

$$u = a\cos(\Omega t + \gamma) + \frac{\epsilon\alpha a^3}{32\Omega^2}\cos(3\Omega t + 3\gamma) + \cdots \tag{5.22}$$

Substituting (5.2) and (5.20) into (5.18) and separating real and imaginary parts, we obtain

$$\dot{a} = -\epsilon\mu a - \frac{\epsilon F}{2\Omega}\sin\gamma \tag{5.23}$$

$$a\dot{\gamma} = -\frac{\epsilon\sigma}{2\Omega}a + \frac{3\epsilon\alpha a^3}{8\Omega} - \frac{\epsilon F}{2\Omega}\cos\gamma \tag{5.24}$$

5.2
Subharmonic Resonance of Order One-Third

To express the nearness of Ω to 3ω, we introduce the detuning parameter σ defined as

$$\tfrac{1}{9}\Omega^2 = \omega^2 + \epsilon\sigma \tag{5.25}$$

Moreover, because this resonance is called secondary resonance, we need to order $F\cos\Omega t$ at $O(1)$ so that its influence is accounted for in the transformed equation. Consequently, using (5.3) to replace $F\cos\Omega t$ with $1/2(z + \bar{z})$ and using (5.25) to replace ω^2 with $1/9\Omega^2 - \epsilon\sigma$, we rewrite (5.1) as

$$\ddot{u} + \tfrac{1}{9}\Omega^2 u = \tfrac{1}{2}(z + \bar{z}) - \epsilon\left(2\mu\dot{u} - \sigma u + \alpha u^3\right) \tag{5.26}$$

To express (5.26) in complex-valued form, we let

$$u = \zeta + \bar{\zeta} \quad \text{and} \quad \dot{u} = \tfrac{1}{3} i\Omega\left(\zeta - \bar{\zeta}\right) \tag{5.27}$$

so that

$$\zeta = \tfrac{1}{2}\left(u - \frac{3i}{\Omega}\dot{u}\right) \quad \text{and} \quad \bar{\zeta} = \tfrac{1}{2}\left(u + \frac{3i}{\Omega}\dot{u}\right) \tag{5.28}$$

Substituting (5.27) and (5.28) into (5.26) yields

$$\begin{aligned}\dot{\zeta} &= \frac{1}{3}i\Omega\,\zeta - \frac{3i}{4\Omega}\,(z+\bar{z})\\ &\quad + \frac{3i\epsilon}{2\Omega}\left[\frac{2}{3}i\mu\Omega\,(\zeta-\bar{\zeta}) - \sigma\,(\zeta+\bar{\zeta}) + \alpha\,(\zeta+\bar{\zeta})^3\right]\end{aligned} \tag{5.29}$$

Because the frequency of the response is $1/3\Omega$ and because z and $\bar{z}$ have the frequencies Ω and $-\Omega$, $z+\bar{z}$ is not a resonance term in this case and because it appears at $O(1)$, we first introduce a linear transformation to remove it from (5.29) at $O(1)$. To this end, we let

$$\zeta = \eta + \Gamma_1 z + \Gamma_2 \bar{z} \tag{5.30}$$

in (5.29) and obtain

$$\dot{\eta} = \frac{1}{3}i\Omega\,\eta + \frac{1}{3}i\Omega\,(\Gamma_1 z + \Gamma_2\bar{z}) - \Gamma_1\dot{z} - \Gamma_2\dot{\bar{z}} - \frac{3i}{4\Omega}\,(z+\bar{z}) + O(\epsilon) \tag{5.31}$$

Using (5.4) to eliminate $\dot{z}$ and $\dot{\bar{z}}$ from (5.31) yields

$$\dot{\eta} = \frac{1}{3}i\Omega\,\eta + \frac{1}{3}i\Omega\,(\Gamma_1 z + \Gamma_2\bar{z}) - i\Omega\,\Gamma_1 z + i\Omega\,\Gamma_2\bar{z} - \frac{3i}{4\Omega}\,(z+\bar{z}) + O(\epsilon) \tag{5.32}$$

Next, we choose Γ_1 and Γ_2 to eliminate z and $\bar{z}$ from (5.32) and obtain

$$\Gamma_1 = -\frac{9}{8\Omega^2} \quad \text{and} \quad \Gamma_2 = \frac{9}{16\Omega^2} \tag{5.33}$$

Substituting (5.30) into (5.29) and using (5.4) and (5.33), we obtain

$$\begin{aligned}\dot{\eta} &= \frac{1}{3}i\Omega\,\eta + \frac{3i\epsilon}{2\Omega}\left[\frac{2}{3}i\mu\Omega\left(\eta - \bar{\eta} - \frac{27z}{16\Omega^2} + \frac{27\bar{z}}{16\Omega^2}\right)\right.\\ &\quad \left. -\sigma\left(\eta+\bar{\eta} - \frac{9z}{16\Omega^2} - \frac{9\bar{z}}{16\Omega^2}\right) + \alpha\left(\eta+\bar{\eta} - \frac{9z}{16\Omega^2} - \frac{9\bar{z}}{16\Omega^2}\right)^3\right]\end{aligned} \tag{5.34}$$

To simplify (5.34), we introduce the near-identity transformation

$$\eta = \xi + \epsilon h\,(\xi, \bar{\xi}, z, \bar{z}) \tag{5.35}$$

and obtain

$$\begin{aligned}\dot{\xi} &= \frac{1}{3}i\Omega\,\xi + \frac{1}{3}i\epsilon\Omega\,h - \epsilon\frac{\partial h}{\partial \xi}\dot{\xi} - \epsilon\frac{\partial h}{\partial \bar{\xi}}\dot{\bar{\xi}} - \epsilon\frac{\partial h}{\partial z}\dot{z} - \epsilon\frac{\partial h}{\partial \bar{z}}\dot{\bar{z}} \\ &+ \frac{3i\epsilon}{2\Omega}\left[\alpha\left(\xi + \bar{\xi} - \frac{9z}{16\Omega^2} - \frac{9\bar{z}}{16\Omega^2}\right)^3\right. \\ &-\sigma\left(\xi + \bar{\xi} - \frac{9z}{16\Omega^2} - \frac{9\bar{z}}{16\Omega^2}\right) \\ &\left.+\frac{2}{3}i\mu\Omega\left(\xi - \bar{\xi} - \frac{27z}{16\Omega^2} + \frac{27\bar{z}}{16\Omega^2}\right)\right] + \cdots \end{aligned} \tag{5.36}$$

Substituting (5.35) into (5.30) and then substituting the result into (5.27), we have

$$u = \xi + \bar{\xi} - \frac{9}{16\Omega^2}(z + \bar{z}) + \epsilon\left(h + \bar{h}\right) + \cdots \tag{5.37}$$

on account of (5.33). To be strictly consistent, we should not include the term $\epsilon(h + \bar{h})$ in (5.37) if we are not going to keep the resonance terms in (5.36) that arise from h.

The form of the $O(\epsilon)$ terms in (5.36) suggests choosing h in the form

$$\begin{aligned}h &= \Delta_1\xi + \Delta_2\bar{\xi} + \Delta_3 z + \Delta_4\bar{z} + \Lambda_1\xi^3 + \Lambda_2\xi^2\bar{\xi} + \Lambda_3\xi\bar{\xi}^2 + \Lambda_4\bar{\xi}^3 \\ &+ \Lambda_5 z^3 + \Lambda_6 z^2\bar{z} + \Lambda_7 z\bar{z}^2 + \Lambda_8\bar{z}^3 + \Lambda_9\xi^2 z + \Lambda_{10}\xi^2\bar{z} + \Lambda_{11}\bar{\xi}^2 z \\ &+ \Lambda_{12}\bar{\xi}^2\bar{z} + \Lambda_{13}\xi z^2 + \Lambda_{14}\bar{\xi} z^2 + \Lambda_{15}\xi\bar{z}^2 + \Lambda_{16}\bar{\xi}\bar{z}^2 + \Lambda_{17}\xi z\bar{z} \\ &+ \Lambda_{18}\bar{\xi} z\bar{z} + \Lambda_{19}\xi\bar{\xi} z + \Lambda_{20}\xi\bar{\xi}\bar{z}\end{aligned} \tag{5.38}$$

Because $\dot{z} = i\Omega z$ and, to the first approximation, $\dot{\xi} = 1/3i\Omega\,\xi$, out of all of the terms in (5.36), only the terms involving $\xi, \xi^2\bar{\xi}, \xi z\bar{z}$, and $\bar{\xi}^2 z$ are resonance terms and hence the coefficients $\Delta_1, \Lambda_2, \Lambda_{11}$, and Λ_{17}, are arbitrary. Moreover, as aforementioned, we are not justified in including $\epsilon(h + \bar{h})$ in (5.37) if we are going to eliminate only the resonance terms from (5.36). Substituting (5.38) into (5.36) and using (5.4), we can choose the Λ_m to eliminate all of the terms except the resonance terms. Consequently, we obtain the simple equation

$$\dot{\xi} = \frac{1}{3}i\Omega\,\xi - \epsilon\mu\xi - \frac{3i\epsilon\sigma}{2\Omega}\xi + \frac{3i\epsilon}{2\Omega}\left[3\alpha\xi^2\bar{\xi} + \frac{243\alpha}{128\Omega^4}\xi z\bar{z} - \frac{27\alpha}{16\Omega^2}z\bar{\xi}^2\right] + \cdots \tag{5.39}$$

Substituting (5.2) and the polar form

$$\xi = \tfrac{1}{2}a e^{i\left(\frac{1}{3}\Omega t + \gamma\right)} \tag{5.40}$$

into (5.37) and (5.39) and separating real and imaginary parts, we obtain to the second approximation

$$u = a\cos\left(\frac{1}{3}\Omega t + \gamma\right) - \frac{9F}{8\Omega^2}\cos\Omega t + \cdots \tag{5.41}$$

where

$$\dot{a} = -\epsilon\mu a - \frac{81\epsilon\alpha F}{64\Omega^3}a^2\sin 3\gamma \tag{5.42}$$

$$a\dot{\gamma} = -\frac{3\epsilon\sigma}{2\Omega}a + \frac{729\epsilon F^2}{256\Omega^5}a + \frac{9\epsilon\alpha}{8\Omega}a^3 - \frac{81\epsilon\alpha F}{64\Omega^3}a^2\cos 3\gamma \tag{5.43}$$

5.3 Superharmonic Resonance of Order Three

To express the nearness of ω to 3Ω, we introduce the detuning parameter σ defined according to

$$9\Omega^2 = \omega^2 + \epsilon\sigma \tag{5.44}$$

As in the case of subharmonic resonance of order one-third, we need to order the excitation at $O(1)$ so that the effect of this resonance will occur at the same order as the effects of the damping and nonlinearity. Substituting (5.44) into (5.1), using (5.3), and assuming that $z = O(1)$, we obtain

$$\ddot{u} + 9\Omega^2 u = \tfrac{1}{2}(z + \bar{z}) - \epsilon\left(2\mu\dot{u} - \sigma u + \alpha u^3\right) \tag{5.45}$$

To express (5.45) in complex-valued form, we let

$$u = \zeta + \bar{\zeta} \quad \text{and} \quad \dot{u} = 3i\Omega\left(\zeta - \bar{\zeta}\right) \tag{5.46}$$

so that

$$\zeta = \frac{1}{2}\left(u - \frac{i}{3\Omega}\dot{u}\right) \quad \text{and} \quad \bar{\zeta} = \frac{1}{2}\left(u + \frac{i}{3\Omega}\dot{u}\right) \tag{5.47}$$

With this transformation, (5.45) becomes

$$\begin{aligned}\dot{\zeta} &= 3i\Omega\zeta - \frac{i}{12\Omega}(z + \bar{z}) \\ &\quad + \frac{i\epsilon}{6\Omega}\left[6i\mu\Omega\left(\zeta - \bar{\zeta}\right) - \sigma\left(\zeta + \bar{\zeta}\right) + \alpha\left(\zeta + \bar{\zeta}\right)^3\right]\end{aligned} \tag{5.48}$$

The first step is to introduce a linear transformation to eliminate the term proportional to $z + \bar{z}$ at $O(1)$. To this end, we let

$$\zeta = \eta + \Gamma_1 z + \Gamma_2 \bar{z} \tag{5.49}$$

in (5.48) and obtain

$$\dot{\eta} = 3i\Omega\eta + 3i\Omega\left(\Gamma_1 z + \Gamma_2\bar{z}\right) - \Gamma_1\dot{z} - \Gamma_2\dot{\bar{z}} - \frac{i}{12\Omega}(z + \bar{z}) + O(\epsilon) \tag{5.50}$$

Using (5.4), we express $\dot{z}$ as $i\Omega z$ and $\dot{\bar{z}}$ as $-i\Omega\bar{z}$ in (5.50) and obtain

$$\dot{\eta} = 3i\Omega\eta + \left(2i\Omega\Gamma_1 - \frac{i}{12\Omega}\right)z + \left(4i\Omega\Gamma_2 - \frac{i}{12\Omega}\right)\bar{z} + O(\epsilon) \tag{5.51}$$

We choose Γ_1 and Γ_2 to eliminate the terms involving z and $\bar{z}$ in (5.51); that is, we put

$$\Gamma_1 = \frac{1}{24\Omega^2} \quad \text{and} \quad \Gamma_2 = \frac{1}{48\Omega^2} \tag{5.52}$$

Substituting (5.49) into (5.48) and using (5.4) and (5.52), we obtain

$$\dot{\eta} = 3i\Omega\eta + \frac{i\epsilon}{6\Omega}\left[6i\mu\Omega\left(\eta - \bar{\eta} + \frac{z}{48\Omega^2} - \frac{\bar{z}}{48\Omega^2}\right)\right.$$
$$\left. -\sigma\left(\eta + \bar{\eta} + \frac{z}{16\Omega^2} + \frac{\bar{z}}{16\Omega^2}\right) + \alpha\left(\eta + \bar{\eta} + \frac{z}{16\Omega^2} + \frac{\bar{z}}{16\Omega^2}\right)^3\right] \tag{5.53}$$

To simplify (5.53), we introduce the near-identity transformation

$$\eta = \xi + \epsilon h\left(\xi, \bar{\xi}, z, \bar{z}\right) \tag{5.54}$$

and obtain

$$\dot{\xi} = 3i\Omega\xi + 3i\epsilon\Omega h - \epsilon\frac{\partial h}{\partial \xi}\dot{\xi} - \epsilon\frac{\partial h}{\partial \bar{\xi}}\dot{\bar{\xi}} - \epsilon\frac{\partial h}{\partial z}\dot{z} - \epsilon\frac{\partial h}{\partial \bar{z}}\dot{\bar{z}}$$
$$+ \frac{i\epsilon}{6\Omega}\left[6i\mu\Omega\left(\xi - \bar{\xi} + \frac{z}{48\Omega^2} - \frac{\bar{z}}{48\Omega^2}\right)\right.$$
$$-\sigma\left(\xi + \bar{\xi} + \frac{z}{16\Omega^2} + \frac{\bar{z}}{16\Omega^2}\right)$$
$$\left. +\alpha\left(\xi + \bar{\xi} + \frac{z}{16\Omega^2} + \frac{\bar{z}}{16\Omega^2}\right)^3\right] + \cdots \tag{5.55}$$

Next, we choose h to eliminate the nonresonance terms. As discussed in the preceding section, if we want to stop at second order, we do not need to determine h and all that we need is to keep the resonance terms in (5.55). The result is

$$\dot{\xi} = 3i\Omega\xi - \epsilon\mu\xi - \frac{i\epsilon\sigma}{6\Omega}\xi + \frac{i\epsilon\alpha}{6\Omega}\left(3\xi^2\bar{\xi} + \frac{3z\bar{z}}{128\Omega^4}\xi + \frac{z^3}{4096\Omega^6}\right) \tag{5.56}$$

Substituting (5.54) into (5.49) and then substituting the result into (5.46), we have

$$u = \xi + \bar{\xi} + \frac{1}{16\Omega^2}(z + \bar{z}) + O(\epsilon) \tag{5.57}$$

Expressing ξ in the polar form

$$\xi = \tfrac{1}{2}ae^{i(3\Omega t + \gamma)} \tag{5.58}$$

and using (5.2) to replace z with $Fe^{i\Omega t}$, we rewrite (5.57) as

$$u = a\cos(3\Omega t + \gamma) + \frac{F}{8\Omega^2}\cos\Omega t + \cdots \tag{5.59}$$

Substituting (5.58) and (5.2) into (5.56) and separating real and imaginary parts, we obtain

$$\dot{a} = -\mu a + \frac{\epsilon\alpha F^3}{12\,288\Omega^7}\sin\gamma \tag{5.60}$$

$$a\dot{\gamma} = -\frac{\epsilon\sigma}{6\Omega}a + \frac{\epsilon\alpha F^2}{256\Omega^5}a + \frac{\epsilon\alpha}{8\Omega}a^3 + \frac{\epsilon\alpha F^3}{12\,288\Omega^7}\cos\gamma \tag{5.61}$$

5.4 An Alternate Approach

In the preceding sections, we used the detuning relationships (5.5), (5.25), and (5.44) to replace ω^2 in terms of $\Omega^2, 1/9\Omega^2$, and $9\Omega^2$ in (5.1). Alternatively, we can have ω as it is in the governing equation and watch for near-resonance terms. We describe this approach for the cases of subharmonic and superharmonic resonances of order one-third and three, respectively. In these cases, F is assumed to be $O(1)$. To express (5.1) in complex-valued form we let

$$u = \zeta + \bar{\zeta} \quad \text{and} \quad \dot{u} = i\omega\left(\zeta - \bar{\zeta}\right) \tag{5.62}$$

so that

$$\zeta = \frac{1}{2}\left(u - \frac{i}{\omega}\dot{u}\right) \quad \text{and} \quad \bar{\zeta} = \frac{1}{2}\left(u + \frac{i}{\omega}\dot{u}\right) \tag{5.63}$$

Using (5.3), (5.62), and (5.63) and the fact that $F = O(1)$, we rewrite (5.1) as

$$\dot{\zeta} = i\omega\zeta - \frac{i}{4\omega}(z + \bar{z}) + \frac{i\epsilon}{2\omega}\left[2i\omega\mu\left(\zeta - \bar{\zeta}\right) + \alpha\left(\zeta + \bar{\zeta}\right)^3\right] \tag{5.64}$$

Again, the first step is to introduce a linear transformation to eliminate the term proportional to $z + \bar{z}$. To this end, we substitute (5.49) into (5.64) and obtain

$$\dot{\eta} = i\omega\eta + i\omega\left(\Gamma_1 z + \Gamma_2\bar{z}\right) - \Gamma_1\dot{z} - \Gamma_2\dot{\bar{z}} - \frac{i}{4\omega}(z + \bar{z}) + O(\epsilon) \tag{5.65}$$

Using (5.4) to replace $\dot{z}$ and $\dot{\bar{z}}$ with $i\Omega z$ and $-i\Omega\bar{z}$, we rewrite (5.65) as

$$\dot{\eta} = i\omega\eta + i\left(\omega\Gamma_1 - \Omega\Gamma_1 - \frac{1}{4\omega}\right)z + i\left(\omega\Gamma_2 + \Omega\Gamma_2 - \frac{1}{4\omega}\right)\bar{z} + O(\epsilon) \tag{5.66}$$

Next, we choose Γ_1 and Γ_2 to eliminate the terms involving z and $\bar{z}$ in (5.66); that is,

$$\Gamma_1 = \frac{1}{4\omega(\omega - \Omega)} \quad \text{and} \quad \Gamma_2 = \frac{1}{4\omega(\omega + \Omega)} \tag{5.67}$$

Substituting (5.49) into (5.64) and using (5.67), we obtain

$$\dot{\eta} = i\omega\eta + \frac{i\epsilon}{2\omega}\left\{2i\omega\mu\left[\eta - \bar{\eta} + \frac{\Omega(z-\bar{z})}{2\omega(\omega^2-\Omega^2)}\right]\right.$$
$$\left. + \alpha\left[\eta + \bar{\eta} + \frac{z+\bar{z}}{2(\omega^2-\Omega^2)}\right]^3\right\} \tag{5.68}$$

To simplify (5.68), we use the near-identity transformation (5.54) and obtain

$$\dot{\xi} = i\omega\xi + i\epsilon\omega h - \epsilon\frac{\partial h}{\partial\xi}\dot{\xi} - \epsilon\frac{\partial h}{\partial\bar{\xi}}\dot{\bar{\xi}} - \epsilon\frac{\partial h}{\partial z}\dot{z} - \epsilon\frac{\partial h}{\partial\bar{z}}\dot{\bar{z}}$$
$$+ \frac{i\epsilon}{2\omega}\left\{2i\omega\mu\left[\xi - \bar{\xi} + \frac{\Omega(z-\bar{z})}{2\omega(\omega^2-\Omega^2)}\right]\right.$$
$$\left. + \alpha\left[\xi + \bar{\xi} + \frac{z+\bar{z}}{2(\omega^2-\Omega^2)}\right]^3\right\} + \cdots \tag{5.69}$$

Then, we choose h to eliminate all of the terms in (5.69) except the resonance and near-resonance terms. The resonance terms correspond to the terms that produce secular terms, whereas the near-resonance terms correspond to terms that produce small-divisor terms in the applications of the method of multiple scales (Nayfeh, 1973, 1981) and the Krylov–Bogoliubov–Mitropolsky technique (Bogoliubov and Mitropolsky, 1961). The resonance terms in (5.69) are the terms proportional to $\xi, \xi^2\bar{\xi}$, and $\xi z\bar{z}$. The near-resonance terms depend on the resonance being considered. In the subharmonic-resonance case of order one-third, the term proportional to $z\bar{\xi}^2$ is the near-resonance term, whereas in the superharmonic-resonance case of order three, the term proportional to z^3 is the near-resonance term. Therefore, keeping the resonance and near-resonance terms in (5.69), we obtain

$$\dot{\xi} = i\omega\xi - \epsilon\mu\xi + \frac{i\epsilon\alpha}{2\omega}\left[3\xi^2\bar{\xi} + \frac{3z\bar{z}}{2(\omega^2-\Omega^2)^2}\xi + \frac{3z}{2(\omega^2-\Omega^2)}\bar{\xi}^2\right] \tag{5.70}$$

when $\Omega \approx 3\omega$ (i.e., when there is a subharmonic resonance of order one-third), and

$$\dot{\xi} = i\omega\xi - \epsilon\mu\xi + \frac{i\epsilon\alpha}{2\omega}\left[3\xi^2\bar{\xi} + \frac{3z\bar{z}}{2(\omega^2-\Omega^2)^2}\xi + \frac{z^3}{8(\omega^2-\Omega^2)^3}\right] \tag{5.71}$$

when $\Omega \approx 1/3\omega$ (i.e., when there is a superharmonic resonance of order three).

Again, to second order, we do not need to determine h. Then, substituting (5.54) and (5.49) into (5.62) and using (5.67), we obtain

$$u = \xi + \bar{\xi} + \frac{1}{2(\omega^2-\Omega^2)}(z+\bar{z}) + \cdots \tag{5.72}$$

Next, substituting the polar form

$$\xi = \frac{1}{2}ae^{i(\omega t+\beta)} \tag{5.73}$$

into (5.72) and using the fact that $z = Fe^{i\Omega t}$, we have

$$u = a\cos(\omega t + \beta) + \frac{F}{\omega^2 - \Omega^2}\cos\Omega t + \cdots \tag{5.74}$$

5.4.1 Subharmonic Case

For the subharmonic-resonance case, substituting (5.73) into (5.70), separating real and imaginary parts, and using the fact that $z = Fe^{i\Omega t}$, we obtain

$$\dot{a} = -\epsilon\mu a + \frac{3\epsilon\alpha F a^2}{8\omega(\omega^2 - \Omega^2)}\sin 3\gamma \tag{5.75}$$

$$a\dot{\beta} = \frac{3\epsilon\alpha}{8\omega}a^3 + \frac{3\epsilon\alpha F^2 a}{4\omega(\omega^2 - \Omega^2)^2} + \frac{3\epsilon\alpha F a^2}{8\omega(\omega^2 - \Omega^2)}\cos 3\gamma \tag{5.76}$$

where

$$3\gamma = 3\beta - (\Omega - 3\omega)\,t \tag{5.77}$$

Substituting for β from (5.77) into (5.74) yields

$$u = a\cos\left(\frac{1}{3}\Omega t + \gamma\right) + \frac{F}{\omega^2 - \Omega^2}\cos\Omega t + \cdots \tag{5.78}$$

while substituting for β from (5.77) into (5.76) yields

$$a\dot{\gamma} = -\frac{1}{3}(\Omega - 3\omega)\,a + \frac{3\epsilon\alpha}{8\omega}a^3 + \frac{3\epsilon\alpha F^2 a}{4\omega(\omega^2 - \Omega^2)^2} + \frac{3\epsilon\alpha F a^2}{8\omega(\omega^2 - \Omega^2)}\cos 3\gamma \tag{5.79}$$

Therefore, to the second approximation, u is given by (5.78), where a and γ are given by the autonomous equations (5.75) and (5.79). These equations are in full agreement with those obtained by using the methods of multiple scales and averaging.

To compare the expansion obtained in this section with that obtained in Section 5.2, we solve for ω in terms of Ω from (5.25) and obtain

$$\omega^2 = \frac{1}{9}\Omega^2 - \epsilon\sigma \quad \text{or} \quad \omega = \frac{1}{3}\Omega - \frac{3\epsilon\sigma}{2\Omega} + \cdots \tag{5.80}$$

Substituting for ω from (5.80) into (5.78), (5.75), and (5.79), we obtain (5.41)–(5.43) to $O(\epsilon)$.

5.4.2
Superharmonic Case

In this case, substituting (5.73) and (5.2) into (5.71) and separating real and imaginary parts, we obtain

$$\dot{a} = -\epsilon\mu a + \frac{\epsilon\alpha F^3}{8\omega(\omega^2 - \Omega^2)^3}\sin\gamma \tag{5.81}$$

$$a\dot{\beta} = \frac{3\epsilon\alpha}{8\omega}a^3 + \frac{3\epsilon\alpha F^2 a}{4\omega(\omega^2 - \Omega^2)^2} + \frac{\epsilon\alpha F^3}{8\omega(\omega^2 - \Omega^2)^3}\cos\gamma \tag{5.82}$$

where

$$\gamma = \beta - (3\Omega - \omega)\,t \tag{5.83}$$

Using (5.83) to eliminate β from (5.74) yields

$$u = a\cos(3\Omega t + \gamma) + \frac{F}{\omega^2 - \Omega^2}\cos\Omega t + \cdots \tag{5.84}$$

while using (5.83) to eliminate β from (5.82) yields

$$a\dot{\gamma} = -(3\Omega - \omega)\,a + \frac{3\epsilon\alpha}{8\omega}a^3 + \frac{3\epsilon\alpha F^2 a}{4\omega(\omega^2 - \Omega^2)^2} + \frac{\epsilon\alpha F^3}{8\omega(\omega^2 - \Omega^2)^3}\cos\gamma \tag{5.85}$$

Therefore, to the second approximation, u is given by (5.84) where a and γ are given by the autonomous equations (5.81) and (5.85). This approximation is in full agreement with that obtained by using the methods of multiple scales and averaging.

To compare the expansion obtained in this section with that obtained in Section 5.3, we use (5.44) to solve for ω in terms of Ω and obtain

$$\omega^2 = 9\Omega^2 - \epsilon\sigma \quad \text{or} \quad \omega = 3\Omega - \frac{\epsilon\sigma}{6\Omega} + \cdots \tag{5.86}$$

Using (5.86) to eliminate ω from (5.84), (5.81), and (5.85), we obtain (5.59)–(5.61) to $O(\epsilon)$.

5.5
Exercises

5.5.1 Consider the system

$$\ddot{u} + \omega^2 u = \epsilon\left(\dot{u} - \tfrac{1}{3}\dot{u}^3\right) + F\cos\Omega t$$

Use the method of normal forms to determine an approximation to this system when $\Omega \approx \omega, 3\omega$, and $1/3\omega$.

5.5.2 Use the method of normal forms to determine a second-order approximation to the solution of

$$\ddot{u} + \omega^2 + 2\epsilon\mu\,\dot{u} + \epsilon\,\alpha\,u^2 = \epsilon F \cos \Omega\, t$$

when $\Omega \approx 2\omega$.

6
Forced Oscillations of SDOF Systems

6.1
Introduction

In this chapter, we consider small- but finite-amplitude responses of a single-degree-of-freedom system with quadratic and cubic nonlinearities to a harmonic excitation; specifically, we consider solutions of

$$\ddot{u} + \omega^2 u + 2\mu\dot{u} + \delta u^2 + \alpha u^3 = F\cos\Omega t \tag{6.1}$$

The results of Section 1.5 show that, to the second approximation, the free undamped solution of (6.1) can be expressed as

$$u = a\cos(\hat{\omega}t + \beta) + \frac{\delta a^2}{6\omega^2}\left[\cos(2\hat{\omega}t + 2\beta) - 3\right] + \cdots$$

where a and β are constants and the frequency $\hat{\omega}$ of free oscillations is given by

$$\hat{\omega} = \omega + \left(\frac{3\alpha}{8\omega} - \frac{5\delta^2}{12\omega^3}\right)a^2 + \cdots$$

Consequently, as the oscillation amplitude a increases, the undamped frequency $\hat{\omega}$ of oscillation is shifted from the undamped linear frequency ω of oscillation by

$$\left(\frac{3\alpha}{8\omega} - \frac{5\delta^2}{12\omega^3}\right)a^2$$

To keep track of the different orders of magnitude, we introduce a small nondimensional parameter as a bookkeeping device and scale the quadratic terms at $O(\epsilon)$ and the cubic term at $O(\epsilon^2)$. The resonances produced by the excitation try to make the response very large. But as the response grows, the nonlinearity, which modifies the natural frequency of the system, and the damping, which dissipates part of the input energy, are activated, thereby limiting the response at small but finite amplitude. Because the quadratic and cubic nonlinearities produce a resonance term first at $O(\epsilon^2)$ according to Section 1.5, we scale the damping at $O(\epsilon^2)$. With these scalings, we rewrite (6.1) as

$$\ddot{u} + \omega^2 u = F\cos\Omega t - \epsilon\delta u^2 - \epsilon^2\left(2\mu\dot{u} + \alpha u^3\right) \tag{6.2}$$

The Method of Normal Forms, Second Edition. Ali Hasan Nayfeh

Because the appearance of resonance terms produced by the excitation depends on the excitation magnitude F and the type of resonance, we scale F differently, depending on the resonance being considered. Carrying out a straightforward expansion, one finds the following resonances:

- $\Omega \approx \omega$: Primary or main resonance
- $\Omega \approx 2\omega$: Subharmonic resonance of order one-half
- $\Omega \approx \frac{1}{2}\omega$: Superharmonic resonance of order two
- $\Omega \approx 3\omega$: Subharmonic resonance of order one-third
- $\Omega \approx \frac{1}{3}\omega$: Superharmonic resonance of order three.

To treat these cases and determine the appropriate scaling of F for each resonance, we cast (6.2) in complex-valued form by letting

$$u = \zeta + \bar{\zeta} \quad \text{and} \quad \dot{u} = i\omega\left(\zeta - \bar{\zeta}\right) \tag{6.3}$$

and

$$z = Fe^{i\Omega t} \tag{6.4}$$

so that

$$\dot{z} = i\Omega z \tag{6.5}$$

Using (6.3)–(6.5), we rewrite (6.2) as

$$\dot{\zeta} = i\omega\zeta - \frac{i}{4\omega}\left(z + \bar{z}\right) + \frac{i\epsilon\delta}{2\omega}\left(\zeta + \bar{\zeta}\right)^2 + \frac{i\epsilon^2}{2\omega}\left[2i\mu\omega\left(\zeta - \bar{\zeta}\right) + \alpha\left(\zeta + \bar{\zeta}\right)^3\right] \tag{6.6}$$

6.2
Primary Resonance

In this case $\Omega \approx \omega$ and we need to scale F at $O(\epsilon^2)$ so that the resonance term produced by the excitation appears at the same order as those produced by the damping and the nonlinearity. With this scaling, we rewrite (6.6) as

$$\dot{\zeta} = i\omega\zeta + \frac{i\epsilon\delta}{2\omega}\left(\zeta + \bar{\zeta}\right)^2 - \epsilon^2\mu\left(\zeta - \bar{\zeta}\right) + \frac{i\epsilon^2}{2\omega}\left[\alpha\left(\zeta + \bar{\zeta}\right)^3 - \frac{1}{2}\left(z + \bar{z}\right)\right] \tag{6.7}$$

To determine a second-order uniform expansion of (6.7), we let

$$\zeta = \eta + \epsilon h_1\left(\eta, \bar{\eta}\right) + \epsilon^2 h_2\left(\eta, \bar{\eta}, z, \bar{z}\right) + \cdots \tag{6.8}$$

where

$$\dot{\eta} = i\omega\eta + \epsilon g_1\left(\eta, \bar{\eta}\right) + \epsilon^2 g_2\left(\eta, \bar{\eta}, z, \bar{z}\right) + \cdots \tag{6.9}$$

and the g_i represent resonance and near-resonance terms that cannot be eliminated by nonsingular choices of the h_i. Substituting (6.8) and (6.9) into (6.7) and equating coefficients of like powers of ϵ, we obtain

$$g_1 + i\omega \left(\frac{\partial h_1}{\partial \eta}\eta - \frac{\partial h_1}{\partial \bar{\eta}}\bar{\eta} - h_1 \right) = \frac{i\delta}{2\omega} (\eta + \bar{\eta})^2 \tag{6.10}$$

$$\begin{aligned} g_2 + i\omega &\left(\frac{\partial h_2}{\partial \eta}\eta - \frac{\partial h_2}{\partial \bar{\eta}}\bar{\eta} - h_2 \right) + i\Omega \left(\frac{\partial h_2}{\partial z}z - \frac{\partial h_2}{\partial \bar{z}}\bar{z} \right) \\ &= -\frac{\partial h_1}{\partial \eta} g_1 - \frac{\partial h_1}{\partial \bar{\eta}} \bar{g}_1 - \mu (\eta - \bar{\eta}) + \frac{i\delta}{\omega} (\eta + \bar{\eta}) \left(h_1 + \bar{h}_1 \right) \\ &+ \frac{i}{2\omega} \left[\alpha (\eta + \bar{\eta})^3 - \frac{1}{2} (z + \bar{z}) \right] \end{aligned} \tag{6.11}$$

Because we will proceed to the next order, we need to explicitly determine h_1. The right-hand side of (6.10) suggests seeking h_1 in the form

$$h_1 = \Gamma_1 \eta^2 + \Gamma_2 \eta\bar{\eta} + \Gamma_3 \bar{\eta}^2 \tag{6.12}$$

Substituting (6.12) into (6.10) yields

$$-g_1 - \left(\omega\Gamma_1 - \frac{\delta}{2\omega} \right) \eta^2 + \left(\omega\Gamma_2 + \frac{\delta}{\omega} \right) \eta\bar{\eta} + \left(3\omega\Gamma_3 + \frac{\delta}{2\omega} \right) \bar{\eta}^2 = 0 \tag{6.13}$$

Hence, the terms proportional to η^2, $\eta\bar{\eta}$, and $\bar{\eta}^2$ can be eliminated from (6.13) if

$$\Gamma_1 = \frac{\delta}{2\omega^2}, \quad \Gamma_2 = -\frac{\delta}{\omega^2}, \quad \Gamma_3 = -\frac{\delta}{6\omega^2} \tag{6.14}$$

Then, (6.13) reduces to $g_1 = 0$.

Substituting (6.12) and (6.14) into (6.11) and using the fact that $g_1 = 0$, we obtain

$$\begin{aligned} g_2 + i\omega &\left(\frac{\partial h_2}{\partial \eta}\eta - \frac{\partial h_2}{\partial \bar{\eta}}\bar{\eta} - h_2 \right) + i\Omega \left(\frac{\partial h_2}{\partial z}z - \frac{\partial h_2}{\partial \bar{z}}\bar{z} \right) = -\mu (\eta - \bar{\eta}) \\ &+ \frac{i\delta^2}{3\omega^3} (\eta + \bar{\eta}) \left(\eta^2 + \bar{\eta}^2 - 6\eta\bar{\eta} \right) + \frac{i}{2\omega} \left[\alpha (\eta + \bar{\eta})^3 - \frac{1}{2} (z + \bar{z}) \right] \end{aligned} \tag{6.15}$$

The usual approach for determining the normal form is to choose h_2 to eliminate as many terms as possible from (6.15), thereby leaving g_2 with the remaining terms. Alternatively, we choose g_2 to eliminate the resonance and near-resonance terms in (6.15) knowing that we can always eliminate the nonresonance terms by a proper choice of h_2. Because we are stopping at this order (seeking a second-order approximation), we do not need to solve explicitly for h_2 and all that we need to do is choose g_2 to eliminate the resonance and near-resonance terms in (6.15); that is, we put

$$g_2 = -\mu\eta + \frac{i}{2\omega} \left(3\alpha - \frac{10\delta^2}{3\omega^2} \right) \eta^2\bar{\eta} - \frac{i}{4\omega} z$$

Substituting for g_1 and g_2 into (6.9), we obtain the normal form

$$\dot{\eta} = i\omega\eta - \epsilon^2\mu\eta + \frac{i\epsilon^2}{2\omega}\left(3\alpha - \frac{10\delta^2}{3\omega^2}\right)\eta^2\bar{\eta} - \frac{i\epsilon^2}{4\omega}z \tag{6.16}$$

Substituting (6.12) and (6.14) into (6.8) and then substituting the result into (6.3), we obtain to the second approximation

$$u = \eta + \bar{\eta} + \frac{\epsilon\delta}{3\omega^2}\left(\eta^2 + \bar{\eta}^2 - 6\eta\bar{\eta}\right) + \cdots \tag{6.17}$$

where η is given by (6.16).

Finally, we introduce the polar representation

$$\eta = \tfrac{1}{2}ae^{i(\omega t+\beta)} \tag{6.18}$$

into (6.17) and obtain

$$u = a\cos(\omega t + \beta) + \frac{\epsilon\delta a^2}{6\omega^2}\left[\cos(2\omega t + 2\beta) - 3\right] + \cdots \tag{6.19}$$

Substituting (6.18) and (6.4) into (6.16) and separating real and imaginary parts, we obtain

$$\dot{a} = -\epsilon^2\mu a + \frac{\epsilon^2 F}{2\omega}\sin\left[(\Omega - \omega)t - \beta\right] \tag{6.20}$$

$$a\dot{\beta} = \epsilon^2\left(\frac{3\alpha}{8\omega} - \frac{5\delta^2}{12\omega^3}\right)a^3 - \frac{\epsilon^2 F}{2\omega}\cos\left[(\Omega - \omega)t - \beta\right] \tag{6.21}$$

Equations 6.19–6.21 are the same as those obtained by using the methods of multiple scales and averaging (Nayfeh and Mook, 1979).

6.3
Subharmonic Resonance of Order One-Half

Because z and $\bar{z}$ are nonresonance terms when Ω is away from ω, we first introduce the linear transformation

$$\zeta = \eta + \Delta_1 z + \Delta_2\bar{z} \tag{6.22}$$

into (6.6) and obtain

$$\dot{\eta} = i\omega\eta + i\omega\left(\Delta_1 z + \Delta_2\bar{z}\right) - \Delta_1\dot{z} - \Delta_2\dot{\bar{z}} - \frac{i}{4\omega}(z + \bar{z}) + O(\epsilon) \tag{6.23}$$

Using (6.5) to express $\dot{z}$ and $\dot{\bar{z}}$ as $i\Omega z$ and $-i\Omega \bar{z}$, we rewrite (6.23) as

$$\dot{\eta} = i\omega\eta + i\left[(\omega - \Omega)\Delta_1 - \frac{1}{4\omega}\right]z + i\left[(\omega + \Omega)\Delta_2 - \frac{1}{4\omega}\right]\bar{z} + O(\epsilon) \tag{6.24}$$

We choose Δ_1 and Δ_2 to eliminate the terms involving z and $\bar{z}$; that is,

$$\Delta_1 = \frac{1}{4\omega(\omega - \Omega)} \quad \text{and} \quad \Delta_2 = \frac{1}{4\omega(\omega + \Omega)} \tag{6.25}$$

It follows from (6.25) that the transformation (6.22) is singular or near singular when $\Omega \approx \omega$; that is, z and $\bar{z}$ would be resonance terms in this case. This is the reason we scaled F and hence z and $\bar{z}$ at $O(\epsilon^2)$ in the preceding section. Substituting (6.22) and (6.25) into (6.6) yields

$$\begin{aligned}\dot{\eta} &= i\omega\eta + \frac{i\epsilon\delta}{2\omega}\left[\eta + \bar{\eta} + \frac{z + \bar{z}}{2(\omega^2 - \Omega^2)}\right]^2 - \epsilon^2\mu(\eta - \bar{\eta}) \\ &+ \frac{i\epsilon^2}{2\omega}\left[\frac{2i\mu\Omega(z - \bar{z})}{(\omega^2 - \Omega^2)} + \alpha\left(\eta + \bar{\eta} + \frac{z + \bar{z}}{2(\omega^2 - \Omega^2)}\right)^3\right]\end{aligned} \tag{6.26}$$

When $\Omega \approx 2\omega$, the excitation produces the near-resonance term $z\bar{\eta}$. In order to have this term appear at the same order at which the nonlinear resonance term $\eta^2\bar{\eta}$ appears, we rescale F and hence z as $O(\epsilon)$ and rewrite (6.26) as

$$\begin{aligned}\dot{\eta} &= i\omega\eta + \frac{i\epsilon\delta}{2\omega}(\eta + \bar{\eta})^2 + \frac{i\epsilon^2\delta}{2\omega(\omega^2 - \Omega^2)}(\eta + \bar{\eta})(z + \bar{z}) - \epsilon^2\mu(\eta - \bar{\eta}) \\ &+ \frac{i\epsilon^2\alpha}{2\omega}(\eta + \bar{\eta})^3 + \cdots\end{aligned} \tag{6.27}$$

Because there are no resonance or near-resonance terms at $O(\epsilon)$ in (6.27), we can eliminate the $O(\epsilon)$ terms, as in the preceding section, by using the near-identity transformation

$$\eta = \xi + \frac{\epsilon\delta}{6\omega^2}(3\xi^2 - \bar{\xi}^2 - 6\xi\bar{\xi}) \tag{6.28}$$

Then, (6.27) becomes

$$\begin{aligned}\dot{\xi} &= i\omega\xi + \frac{i\epsilon^2\delta}{2\omega(\omega^2 - \Omega^2)}(\xi + \bar{\xi})(z + \bar{z}) - \epsilon^2\mu(\xi - \bar{\xi}) \\ &+ \frac{i\epsilon^2\alpha}{2\omega}(\xi + \bar{\xi})^3 + \frac{i\epsilon^2\delta^2}{3\omega^3}(\xi + \bar{\xi})(\xi^2 + \bar{\xi}^2 - 6\xi\bar{\xi})\end{aligned} \tag{6.29}$$

To simplify (6.29), we can introduce a near-identify transformation to eliminate the nonresonance terms. Because we are stopping at this order, we need to keep only the resonance terms ξ and $\xi^2\bar{\xi}$ and near-resonance term $z\bar{\xi}$. The result is

$$\dot{\xi} = i\omega\xi - \epsilon^2\mu\xi + \frac{i\epsilon^2}{2\omega}\left(3\alpha - \frac{10\delta^2}{3\omega^2}\right)\xi^2\bar{\xi} + \frac{i\epsilon^2\delta}{2\omega(\omega^2 - \Omega^2)}z\bar{\xi} \tag{6.30}$$

Substituting (6.22) and (6.28) into (6.3), using (6.25), and recalling that F is scaled at $O(\epsilon)$, we obtain to the second approximation

$$u = \xi + \bar{\xi} + \frac{\epsilon(z+\bar{z})}{2(\omega^2 - \Omega^2)} + \frac{\epsilon\delta}{3\omega^2}\left(\xi^2 + \bar{\xi}^2 - 6\xi\bar{\xi}\right) + \cdots \tag{6.31}$$

where ξ is given by (6.30).

Substituting the polar forms (6.4) and (6.18) into (6.31) yields

$$u = a\cos(\omega t + \beta) + \frac{\epsilon F}{\omega^2 - \Omega^2}\cos\Omega t + \frac{\epsilon\delta a^2}{6\omega^2}\left[\cos(2\omega t + 2\beta) - 3\right] + \cdots \tag{6.32}$$

while substituting (6.4) and (6.18) into (6.30) and separating real and imaginary parts yields

$$\dot{a} = -\epsilon^2\mu a + \frac{\epsilon^2\delta F}{2\omega(\omega^2 - \Omega^2)}\sin\left[2\beta - (\Omega - 2\omega)t\right] \tag{6.33}$$

$$a\dot{\beta} = \epsilon^2\left(\frac{3\alpha}{8\omega} - \frac{5\delta^2}{12\omega^3}\right)a^3 + \frac{\epsilon^2\delta F}{2\omega(\omega^2 - \Omega^2)}\cos\left[2\beta - (\Omega - 2\omega)t\right] \tag{6.34}$$

Equations 6.32–6.34 are in full agreement with those obtained by using the method of multiple scales (Nayfeh, 1986).

6.4 Superharmonic Resonance of Order Two

In this case, $\Omega \approx 1/2\omega$. Inspection of (6.26) reveals that the resonance term arising from the excitation is z^2. In order that its influence balance the resonance terms arising from the damping and the nonlinearity, we scale F and hence z at $O(\epsilon^{1/2})$ and then rewrite (6.26) as

$$\begin{aligned}\dot{\eta} = i\omega\eta &+ \frac{i\epsilon\delta}{2\omega}(\eta + \bar{\eta})^2 + \frac{i\epsilon^{3/2}\delta}{2\omega(\omega^2 - \Omega^2)}(\eta + \bar{\eta})(z + \bar{z}) - \epsilon^2\mu(\eta - \bar{\eta}) \\ &+ \frac{i\epsilon^2\delta(z+\bar{z})^2}{8\omega(\omega^2 - \Omega^2)^2} + \frac{i\epsilon^2\alpha}{2\omega}(\eta + \bar{\eta})^3 + \cdots\end{aligned} \tag{6.35}$$

As in the preceding section, we introduce the near-identity transformation (6.28) to eliminate the $O(\epsilon)$ terms in (6.35) and hence obtain

$$\begin{aligned}\dot{\xi} = i\omega\xi &+ \frac{i\epsilon^{3/2}\delta}{2\omega(\omega^2 - \Omega^2)}(\xi + \bar{\xi})(z + \bar{z}) - \epsilon^2\mu(\xi - \bar{\xi}) \\ &+ \frac{i\epsilon^2\delta(z+\bar{z})^2}{8\omega(\omega^2 - \Omega^2)^2} + \frac{i\epsilon^2\delta^2}{3\omega^3}(\xi + \bar{\xi})(\xi^2 + \bar{\xi}^2 - 6\xi\bar{\xi}) \\ &+ \frac{i\epsilon^2\alpha}{2\omega}(\xi + \bar{\xi})^3 + \cdots\end{aligned} \tag{6.36}$$

Because there are no resonance terms at $O(\epsilon^{3/2})$, we can introduce a near-identity transformation

$$\xi = \chi + \epsilon^{3/2} h\,(\chi, \bar{\chi}, z, \bar{z}) \tag{6.37}$$

to eliminate these terms and rewrite (6.36) as

$$\begin{aligned}\dot{\chi} &= i\omega\chi + \frac{i\epsilon^2\delta^2}{3\omega^3}\,(\chi + \bar{\chi})\left(\chi^2 + \bar{\chi}^2 - 6\chi\bar{\chi}\right) - \epsilon^2\mu\,(\chi - \bar{\chi}) \\ &\quad + \frac{i\epsilon^2\alpha}{2\omega}\,(\chi + \bar{\chi})^3 + \frac{i\epsilon^2\delta}{8\omega(\omega^2 - \Omega^2)^2}\,(z + \bar{z})^2 + \cdots\end{aligned} \tag{6.38}$$

Because our aim is to obtain an expression that is valid to $O(\epsilon^2)$, we do not need to determine h. Moreover, we also know that we can choose a near-identity transformation

$$\chi = w + \epsilon^2 g\,(w, \bar{w}, z, \bar{z}) \tag{6.39}$$

to eliminate the nonresonance terms from (6.38) and obtain

$$\dot{w} = i\omega w - \epsilon^2\mu w + \frac{i\epsilon^2}{2\omega}\left(3\alpha - \frac{10\delta^2}{3\omega^2}\right) w^2\bar{w} + \frac{i\epsilon^2\delta}{8\omega(\omega^2 - \Omega^2)^2} z^2 + \cdots \tag{6.40}$$

Again, for an expression that is valid to $O(\epsilon^2)$, we do not need to determine g. Substituting (6.22), (6.25), (6.28), (6.37), and (6.39) into (6.3), we have

$$u = w + \bar{w} + \frac{\epsilon^{1/2}(z + \bar{z})}{2(\omega^2 - \Omega^2)} + \frac{\epsilon\delta}{3\omega^2}\left(w^2 + \bar{w}^2 - 6w\bar{w}\right) + \cdots \tag{6.41}$$

where use has been made of the assumption that F and hence z are $O(\epsilon^{1/2})$.

Expressing w in the polar form

$$w = \frac{1}{2} a e^{i(\omega t + \beta)} \tag{6.42}$$

we rewrite (6.41) as

$$u = a\cos(\omega t + \beta) + \frac{\epsilon^{1/2} F}{\omega^2 - \Omega^2}\cos\Omega t + \frac{\epsilon\delta a^2}{6\omega^2}\left[\cos(2\omega t + 2\beta) - 3\right] + \cdots \tag{6.43}$$

Substituting (6.4) and (6.42) into (6.40) and separating real and imaginary parts, we obtain

$$\dot{a} = -\epsilon^2\mu a + \frac{\epsilon^2\delta F^2}{4\omega(\omega^2 - \Omega^2)^2}\sin\left[\beta - (2\Omega - \omega)\,t\right] \tag{6.44}$$

$$a\dot{\beta} = \epsilon^2\left(\frac{3\alpha}{8\omega} - \frac{5\delta^2}{12\omega^3}\right) a^3 + \frac{\epsilon^2\delta F^2}{4\omega(\omega^2 - \Omega^2)^2}\cos\left[\beta - (2\Omega - \omega)\,t\right] \tag{6.45}$$

Equations 6.43–6.45 are in full agreement with those obtained by using the method of multiple scales (Nayfeh, 1986).

6.5
Subharmonic Resonance of Order One-Third

In this case, $\Omega \approx 3\omega$. Inspection of (6.26) shows that the resonance term produced by the excitation is $z\bar{\eta}^2$. Because this resonance term occurs at $O(\epsilon^2)$, F and hence z are $O(1)$. Because none of the $O(\epsilon)$ terms is resonance in this case, one can introduce a near-identity transformation in the form

$$\eta = \xi + \epsilon g\left(\xi, \bar{\xi}, z, \bar{z}\right) \tag{6.46}$$

and eliminate them. The form of the $O(\epsilon)$ terms suggests seeking g in the form

$$\begin{aligned} g = {} & \Lambda_1\xi^2 + \Lambda_2\xi\bar{\xi} + \Lambda_3\bar{\xi}^2 + \Lambda_4 z^2 + \Lambda_5 z\bar{z} + \Lambda_6\bar{z}^2 + \Lambda_7\xi z + \Lambda_8\xi\bar{z} \\ & + \Lambda_9\bar{\xi}z + \Lambda_{10}\bar{\xi}\bar{z} \end{aligned} \tag{6.47}$$

Substituting (6.46) into (6.26) shows that

$$\dot{\xi} = i\omega\xi + O(\epsilon) \tag{6.48}$$

Consequently, substituting (6.46) and (6.47) into (6.26) and replacing $\dot{\bar{z}}$ and $\dot{z}$ with $-i\Omega\bar{z}$ and $i\Omega z$ and $\dot{\bar{\xi}}$ and $\dot{\xi}$ with $-i\omega\bar{\xi}$ and $i\omega\xi$ on the right-hand side of the resulting equation, we obtain to $O(\epsilon)$

$$\begin{aligned} \dot{\xi} = {} & i\omega\xi + i\epsilon\left[-\omega\Lambda_1 + \frac{\delta}{2\omega}\right]\xi^2 + i\epsilon\left[\omega\Lambda_2 + \frac{\delta}{\omega}\right]\xi\bar{\xi} \\ & + i\epsilon\left[3\omega\Lambda_3 + \frac{\delta}{2\omega}\right]\bar{\xi}^2 + i\epsilon\left[\omega\Lambda_5 + \frac{\delta}{4\omega(\omega^2-\Omega^2)^2}\right]z\bar{z} \\ & + i\epsilon\left[(\omega - 2\Omega)\Lambda_4 + \frac{\delta}{8\omega(\omega^2-\Omega^2)^2}\right]z^2 \\ & + i\epsilon\left[(\omega + 2\Omega)\Lambda_6 + \frac{\delta}{8\omega(\omega^2-\Omega^2)^2}\right]\bar{z}^2 \\ & - i\epsilon\left[\Omega\Lambda_7 - \frac{\delta}{2\omega(\omega^2-\Omega^2)}\right]\xi z \\ & + i\epsilon\left[\Omega\Lambda_8 + \frac{\delta}{2\omega(\omega^2-\Omega^2)}\right]\xi\bar{z} \\ & + i\epsilon\left[(2\omega - \Omega)\Lambda_9 + \frac{\delta}{2\omega(\omega^2-\Omega^2)}\right]\bar{\xi}z \\ & + i\epsilon\left[(2\omega + \Omega)\Lambda_{10} + \frac{\delta}{2\omega(\omega^2-\Omega^2)}\right]\bar{\xi}\bar{z} + O(\epsilon^2) \end{aligned} \tag{6.49}$$

Choosing the Λ_m to eliminate all of the $O(\epsilon)$ terms in (6.49), we have

$$\begin{aligned}
\Lambda_1 &= \frac{\delta}{2\omega^2}, \quad \Lambda_2 = -\frac{\delta}{\omega^2}, \quad \Lambda_3 = -\frac{\delta}{6\omega^2}, \\
\Lambda_4 &= -\frac{\delta}{8\omega(\omega^2-\Omega^2)^2(\omega-2\Omega)}, \\
\Lambda_5 &= -\frac{\delta}{4\omega^2(\omega^2-\Omega^2)^2}, \quad \Lambda_6 = -\frac{\delta}{8\omega(\omega^2-\Omega^2)^2(\omega+2\Omega)}, \\
\Lambda_7 &= \frac{\delta}{2\omega\Omega(\omega^2-\Omega^2)}, \quad \Lambda_8 = -\frac{\delta}{2\omega\Omega(\omega^2-\Omega^2)}, \\
\Lambda_9 &= -\frac{\delta}{2\omega(\omega^2-\Omega^2)(2\omega-\Omega)}, \quad \Lambda_{10} = -\frac{\delta}{2\omega(\omega^2-\Omega^2)(2\omega+\Omega)}
\end{aligned} \tag{6.50}$$

It is clear from (6.50) that, in addition to the small-divisor term that occurs when $\Omega \approx \omega$ (primary resonance), there are small-divisor terms when $\Omega \approx 2\omega$ (subharmonic resonance of order one-half) and $\Omega \approx 1/2\omega$ (superharmonic resonance of order two). Thus, the terms involving $z\bar{\xi}$ and z^2 are near-resonance terms. These are the cases treated in the preceding two sections.

Substituting (6.46) into (6.26) and using (6.47) and (6.50) yields

$$\begin{aligned}
\dot{\xi} = i\omega\xi &+ \frac{i\epsilon^2\delta^2}{\omega}\left[\xi+\bar{\xi}+\frac{z+\bar{z}}{2(\omega^2-\Omega^2)}\right]\left[\frac{1}{3\omega^2}\left(\xi^2+\bar{\xi}^2-6\xi\bar{\xi}\right)\right. \\
&- \frac{z^2+\bar{z}^2}{4(\omega^2-\Omega^2)^2(\omega^2-4\Omega^2)} - \frac{z\bar{z}}{2\omega^2(\omega^2-\Omega^2)^2} \\
&\left.+ \frac{\xi z+\bar{\xi}\bar{z}}{\Omega(2\omega+\Omega)(\omega^2-\Omega^2)} - \frac{\xi\bar{z}+\bar{\xi}z}{\Omega(2\omega-\Omega)(\omega^2-\Omega^2)}\right] \\
&+ \frac{i\epsilon^2}{2\omega}\left[2i\mu\omega\left(\xi-\bar{\xi}+\frac{\Omega(z-\bar{z})}{2\omega(\omega^2-\Omega^2)}\right)\right. \\
&\left.+ \alpha\left(\xi+\bar{\xi}+\frac{z+\bar{z}}{2(\omega^2-\Omega^2)}\right)^3\right] + \cdots
\end{aligned} \tag{6.51}$$

Next, we introduce the near-identity transformation

$$\xi = w + \epsilon^2 h(w, \bar{w}, z, \bar{z}) \tag{6.52}$$

to eliminate the nonresonance terms from (6.51). The remainder depends on the type of resonance being considered. Because we are stopping at this order, we do not need to determine h. In the case of subharmonic resonance of order one-third, the excitation produces the near-resonance term $z\bar{w}^2$. Keeping this term as well as the resonance terms in the transformed equation (6.51), we obtain

$$\begin{aligned}\dot{w} &= i\omega w - \epsilon^2 \mu w + \frac{i\epsilon^2}{2\omega}\left(3\alpha - \frac{10\delta^2}{3\omega^2}\right) w^2 \bar{w} \\ &+ \frac{i\epsilon^2}{\omega(\omega^2-\Omega^2)^2}\left[\frac{3}{4}\alpha - \delta^2\left(\frac{1}{2\omega^2} + \frac{1}{4\omega^2-\Omega^2}\right)\right] z\bar{z}w \\ &+ \frac{i\epsilon^2}{\omega(\omega^2-\Omega^2)}\left[\frac{3}{4}\alpha + \delta^2\left(\frac{1}{6\omega^2} - \frac{1}{\Omega(2\omega-\Omega)}\right)\right] z\bar{w}^2 \end{aligned} \tag{6.53}$$

In the case of superharmonic resonance of order three, the excitation produces the near-resonance term z^3. Then, keeping this term as well as the resonance terms in the transformed equation (6.51), we obtain

$$\begin{aligned}\dot{w} &= i\omega w - \epsilon^2 \mu w + \frac{i\epsilon^2}{2\omega}\left(3\alpha - \frac{10\delta^2}{3\omega^2}\right) w^2 \bar{w} \\ &+ \frac{i\epsilon^2}{\omega(\omega^2-\Omega^2)^2}\left[\frac{3}{4}\alpha - \delta^2\left(\frac{1}{2\omega^2} + \frac{1}{4\omega^2-\Omega^2}\right)\right] z\bar{z}w \\ &+ \frac{i\epsilon^2}{16\omega(\omega^2-\Omega^2)^3}\left(\alpha - \frac{2\delta^2}{\omega^2-4\Omega^2}\right) z^3 \end{aligned} \tag{6.54}$$

Substituting the transformations (6.52), (6.46), and (6.22) into (6.3), using the expressions (6.25), (6.47), and (6.50), and recalling that $z = O(1)$, we obtain

$$\begin{aligned}u &= w + \bar{w} + \frac{z+\bar{z}}{2(\omega^2-\Omega^2)} + \epsilon\left[\frac{\delta}{3\omega^2}\left(w^2 + \bar{w}^2 - 6w\bar{w}\right)\right. \\ &- \frac{z^2+\bar{z}^2}{4(\omega^2-\Omega^2)^2(\omega^2-4\Omega^2)} - \frac{z\bar{z}}{2\omega^2(\omega^2-\Omega^2)^2} \\ &\left. + \frac{wz+\bar{w}\bar{z}}{\Omega(2\omega+\Omega)(\omega^2-\Omega^2)} - \frac{w\bar{z}+\bar{w}z}{\Omega(2\omega-\Omega)(\omega^2-\Omega^2)}\right] + \cdots \end{aligned} \tag{6.55}$$

Substituting (6.4) and the polar form (6.42) into (6.55) yields

$$\begin{aligned}u &= a\cos(\omega t + \beta) + \frac{F}{\omega^2-\Omega^2}\cos\Omega t + \epsilon\left\{-\frac{F^2}{2\omega^2(\omega^2-\Omega^2)^2}\right. \\ &- \frac{F^2\cos 2\Omega t}{2(\omega^2-\Omega^2)^2(\omega^2-4\Omega^2)} + \frac{\delta a^2}{6\omega^2}\left[\cos(2\omega t + 2\beta) - 3\right] \\ &\left. + \frac{Fa\cos\left[(\Omega+\omega)t+\beta\right]}{\Omega(2\omega+\Omega)(\omega^2-\Omega^2)} - \frac{Fa\cos\left[(\Omega-\omega)t-\beta\right]}{\Omega(2\omega-\Omega)(\omega^2-\Omega^2)}\right\} + \cdots \end{aligned} \tag{6.56}$$

Substituting (6.4) and (6.42) into (6.53) and separating real and imaginary parts, we have

$$\dot{a} = -\epsilon^2 \mu a + \frac{\epsilon^2}{2\omega(\omega^2 - \Omega^2)} \left[\frac{3}{4}\alpha + \delta^2 \left(\frac{1}{6\omega^2} - \frac{1}{\Omega(2\omega - \Omega)}\right)\right] \bullet Fa^2 \sin[3\beta - (\Omega - 3\omega)t] \tag{6.57}$$

$$a\dot{\beta} = \frac{\epsilon^2 F^2}{\omega(\omega^2 - \Omega^2)^2} \left[\frac{3}{4}\alpha - \delta^2 \left(\frac{1}{2\omega^2} + \frac{1}{4\omega^2 - \Omega^2}\right)\right] a + \epsilon^2 \left(\frac{3\alpha}{8\omega} - \frac{5\delta^2}{12\omega^3}\right) a^3 + \frac{\epsilon^2}{2\omega(\omega^2 - \Omega^2)} \cdot \left[\frac{3}{4}\alpha + \delta^2 \left(\frac{1}{6\omega^2} - \frac{1}{\Omega(2\omega - \Omega)}\right)\right] Fa^2 \cos[3\beta - (\Omega - 3\omega)t] \tag{6.58}$$

for the subharmonic case. Equations 6.56–6.58 are in full agreement with those obtained by using the method of multiple scales (Nayfeh, 1986).

In the case of superharmonic resonance of order three, substituting (6.4) and (6.42) into (6.54) and separating real and imaginary parts, we obtain

$$\dot{a} = -\epsilon^2 \mu a + \frac{\epsilon^2 F^3}{8\omega(\omega^2 - \Omega^2)^3} \left(\alpha - \frac{2\delta^2}{\omega^2 - 4\Omega^2}\right) \sin[\beta - (3\Omega - \omega)t] \tag{6.59}$$

$$a\dot{\beta} = \frac{\epsilon^2 F^2}{\omega(\omega^2 - \Omega^2)^2} \left[\frac{3}{4}\alpha - \delta^2 \left(\frac{1}{2\omega^2} + \frac{1}{4\omega^2 - \Omega^2}\right)\right] a + \epsilon^2 \left(\frac{3\alpha}{8\omega} - \frac{5\delta^2}{12\omega^3}\right) a^3 + \frac{\epsilon^2 F^3}{8\omega(\omega^2 - \Omega^2)^3} \left(\alpha - \frac{2\delta^2}{\omega^2 - 4\Omega^2}\right) \cos[\beta - (3\Omega - \omega)t] \tag{6.60}$$

Equations 6.56, 6.59, and 6.60 are in full agreement with those obtained by using the method of multiple scales (Nayfeh, 1986).

7
Parametrically Excited Systems

In the preceding two chapters, the excitation was taken to be external or sometimes called additive. In this chapter, we treat the case in which the excitation is parametric or sometimes called multiplicative. We start with the Mathieu equation and then progress to linear multiple-degree-of-freedom systems and finally include the effect of nonlinearities.

7.1
The Mathieu Equation

In this section, we determine approximations to the solutions of the Mathieu equation

$$\ddot{u} + \omega^2 u + 2\epsilon\mu\dot{u} + 2\epsilon u \cos \Omega t = 0 \tag{7.1}$$

where ϵ is a small nondimensional parameter and μ, ω, and Ω are constants.

As in the preceding chapters, we first cast (7.1) into a complex-valued form by introducing the transformation

$$u = \zeta + \bar{\zeta} \quad \text{and} \quad \dot{u} = i\omega\left(\zeta - \bar{\zeta}\right) \tag{7.2}$$

and

$$2\cos \Omega t = z + \bar{z}\,, \quad z = e^{i\Omega t} \tag{7.3}$$

so that

$$\dot{z} = i\Omega z \tag{7.4}$$

With this transformation, (7.1) becomes

$$\dot{\zeta} = i\omega\zeta - \epsilon\mu\left(\zeta - \bar{\zeta}\right) + \frac{i\epsilon}{2\omega}\left(\zeta + \bar{\zeta}\right)(z + \bar{z}) \tag{7.5}$$

We note that the transformation (7.2) is not valid when $\omega \approx 0$. This case is treated in Section 2.8.

The Method of Normal Forms, Second Edition. Ali Hasan Nayfeh

To determine a second-order uniform expansion of and hence determine a normal form of (7.5), we let

$$\zeta = \eta + \epsilon h_1 (\eta, \bar{\eta}, z, \bar{z}) + \epsilon^2 h_2 (\eta, \bar{\eta}, z, \bar{z}) + \cdots \tag{7.6}$$

where

$$\dot{\eta} = i\omega\eta + \epsilon g_1 (\eta, \bar{\eta}, z, \bar{z}) + \epsilon^2 g_2 (\eta, \bar{\eta}, z, \bar{z}) + \cdots \tag{7.7}$$

Substituting (7.6) and (7.7) into (7.5) and equating coefficients of like powers of ϵ, we obtain

$$g_1 + \mathcal{L}(h_1) = -\mu (\eta - \bar{\eta}) + \frac{i}{2\omega} (\eta + \bar{\eta}) (z + \bar{z}) \tag{7.8}$$

$$g_2 + \mathcal{L}(h_2) = -\frac{\partial h_1}{\partial \eta} g_1 - \frac{\partial h_1}{\partial \bar{\eta}} \bar{g}_1 - \mu \left(h_1 - \bar{h}_1\right) + \frac{i}{2\omega} \left(h_1 + \bar{h}_1\right) (z + \bar{z}) \tag{7.9}$$

where the operator $\mathcal{L}$ is defined by

$$\mathcal{L} = i\omega \left(\eta \frac{\partial}{\partial \eta} - \bar{\eta} \frac{\partial}{\partial \bar{\eta}} - 1\right) + i\Omega \left(z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}}\right) \tag{7.10}$$

Next, we consider two cases: fundamental parametric resonance (i.e., $\Omega \approx \omega$) and principal parametric resonance (i.e., $\Omega \approx 2\omega$).

7.1.1
Fundamental Parametric Resonance

We start by choosing g_1 to eliminate the resonance and near-resonance terms in (7.8). Because $\Omega \approx \omega$, only the term $-\mu\eta$ is a resonance term and there are no near-resonance terms. Therefore, we choose g_1 to eliminate this resonance term; that is,

$$g_1 = -\mu\eta \tag{7.11}$$

Then, we choose h_1 to eliminate all of the nonresonance terms in (7.8). The right-hand side of (7.8) suggests seeking h_1 in the form

$$h_1 = \Gamma_1 \bar{\eta} + \Gamma_2 \eta z + \Gamma_3 \bar{\eta} \bar{z} + \Gamma_4 \eta \bar{z} + \Gamma_5 \bar{\eta} z \tag{7.12}$$

Substituting (7.12) into (7.8) and using (7.11) yields

$$(2i\omega\Gamma_1 + \mu)\bar{\eta} + i\left(-\Omega\Gamma_2 + \frac{1}{2\omega}\right)\eta z + i\left(\Omega\Gamma_4 + \frac{1}{2\omega}\right)\eta\bar{z} + i\left((2\omega + \Omega)\Gamma_3 + \frac{1}{2\omega}\right)\bar{\eta}\bar{z} + i\left((2\omega - \Omega)\Gamma_5 + \frac{1}{2\omega}\right)\bar{\eta} z = 0 \tag{7.13}$$

Equating each of the coefficients of $\bar{\eta}$, ηz, $\bar{\eta}\bar{z}$, $\eta\bar{z}$, and $\bar{\eta}z$ to zero, we obtain

$$\Gamma_1 = \frac{i\mu}{2\omega}\,, \quad \Gamma_2 = \frac{1}{2\omega\Omega}\,, \quad \Gamma_3 = -\frac{1}{2\omega(\Omega+2\omega)}\,,$$
$$\Gamma_4 = -\frac{1}{2\omega\Omega}\,, \quad \Gamma_5 = \frac{1}{2\omega(\Omega-2\omega)} \tag{7.14}$$

Substituting (7.11) and (7.12) into (7.9) and using (7.14), we have

$$g_2 + \mathcal{L}(h_2) = -\mu\left[\frac{i\mu}{2\omega}\eta - \frac{\Omega\eta z}{4\omega^2(\Omega+2\omega)} + \frac{(\Omega-2\omega)\bar{\eta}\bar{z}}{4\omega^2\Omega} - \frac{\Omega\eta\bar{z}}{4\omega^2(\Omega-2\omega)}\right.$$
$$\left. + \frac{(\Omega+2\omega)\bar{\eta}z}{4\omega^2\Omega}\right] + \frac{i}{2\omega}\left[\frac{\eta z+\bar{\eta}\bar{z}}{\Omega(\Omega+2\omega)} + \frac{\eta\bar{z}+\bar{\eta}z}{\Omega(\Omega-2\omega)}\right](z+\bar{z}) \tag{7.15}$$

We note that the terms proportional to η and $\eta z\bar{z}$ are resonance terms and the term proportional to $\bar{\eta}z^2$ is a near-resonance term because $\Omega \approx \omega$. Hence, choosing g_2 to eliminate these resonance and near-resonance terms, we have

$$g_2 = -\frac{i\mu^2}{2\omega}\eta + \frac{i}{2\omega}\left[\frac{2\eta z\bar{z}}{\Omega^2-4\omega^2} + \frac{z^2\bar{\eta}}{\Omega(\Omega-2\omega)}\right] \tag{7.16}$$

Substituting for g_1 and g_2 from (7.11) and (7.16) into (7.7), we obtain the normal form

$$\dot{\eta} = i\omega\eta - \epsilon\mu\eta - \frac{i\epsilon^2\mu^2}{2\omega}\eta + \frac{i\epsilon^2}{2\omega}\left[\frac{2\eta z\bar{z}}{\Omega^2-4\omega^2} + \frac{z^2\bar{\eta}}{\Omega(\Omega-2\omega)}\right] \tag{7.17}$$

Substituting (7.6) and (7.12) into (7.2) and using (7.14), we obtain

$$u = \eta+\bar{\eta}+\epsilon\left[-\frac{i\mu}{2\omega}\left(\eta-\bar{\eta}\right) + \frac{\eta z+\bar{\eta}\bar{z}}{\Omega(\Omega+2\omega)} + \frac{\eta\bar{z}+\bar{\eta}z}{\Omega(\Omega-2\omega)}\right]+\cdots \tag{7.18}$$

Expressing η in the polar form

$$\eta = \frac{1}{2}a^{i(\Omega t+\beta)} \tag{7.19}$$

and using (7.3), we rewrite (7.18) as

$$u = a\cos(\Omega t+\beta) + \epsilon\left[\frac{\mu a\sin(\Omega t+\beta)}{2\omega} + \frac{a\cos(2\Omega t+\beta)}{\Omega(\Omega+2\omega)}\right.$$
$$\left. + \frac{a\cos(\beta)}{\Omega(\Omega-2\omega)}\right] + \cdots \tag{7.20}$$

Substituting (7.3) and (7.19) into (7.17) and separating real and imaginary parts, we obtain

$$\dot{a} = -\epsilon\mu a + \frac{\epsilon^2 a\sin(2\beta)}{2\omega\Omega(\Omega-2\omega)} \tag{7.21}$$

$$a\dot{\beta} = -(\Omega-\omega)a - \frac{\epsilon^2\mu^2}{2\omega}a + \frac{\epsilon^2 a}{\omega(\Omega^2-4\omega^2)} + \frac{\epsilon^2 a\cos(2\beta)}{2\omega\Omega(\Omega-2\omega)} \tag{7.22}$$

7.1.2 Principal Parametric Resonance

Because $\Omega \approx 2\omega$ in this case, $z\bar{\eta}$ is a near-resonance term. Consequently, choosing g_1 to eliminate the resonance and near-resonance terms in (7.8), we obtain

$$g_1 = -\mu\eta + \frac{i}{2\omega}\bar{\eta}z \tag{7.23}$$

Then, we seek the solution of (7.8) in the form

$$h_1 = \Gamma_1\bar{\eta} + \Gamma_2\eta z + \Gamma_3\bar{\eta}\bar{z} + \Gamma_4\eta\bar{z} \tag{7.24}$$

Following steps similar to those used in the preceding section, we find that the Γ_i for $i = 1, 2, 3, 4$ are given by (7.14).

Substituting (7.23) and (7.24) into (7.9), using (7.14), and choosing g_2 to eliminate the resonance and near-resonance terms, we obtain

$$g_2 = -\frac{i\mu^2}{2\omega}\eta - \frac{i}{4\omega^2(\Omega + 2\omega)}\eta z\bar{z} - \frac{\mu(\Omega + 2\omega)}{4\omega^2\Omega}z\bar{\eta} \tag{7.25}$$

Substituting (7.23) and (7.25) into (7.7) yields the normal form

$$\dot{\eta} = i\omega\eta - \epsilon\left[\mu\eta - \frac{iz\bar{\eta}}{2\omega}\right] - \epsilon^2\left[\frac{i\mu^2\eta}{2\omega} + \frac{iz\bar{z}\eta}{4\omega^2(\Omega + 2\omega)} + \frac{\mu(\Omega+2\omega)z\bar{\eta}}{4\omega^2\Omega}\right] \tag{7.26}$$

Substituting (7.6) into (7.12) and using (7.24) and (7.14), we have

$$u = \eta + \bar{\eta} + \epsilon\left[-\frac{i\mu}{2\omega}(\eta - \bar{\eta}) + \frac{\eta z + \bar{\eta}\bar{z}}{\Omega(\Omega + 2\omega)} - \frac{\eta\bar{z} + \bar{\eta}z}{2\Omega\omega}\right] + \cdots \tag{7.27}$$

Because $z = e^{i\Omega t}$, the form of the near-resonance term in (7.26) suggests the following polar form for η:

$$\eta = \frac{1}{2}a e^{i\left(\frac{1}{2}\Omega t + \beta\right)} \tag{7.28}$$

We note that this choice produces a set of autonomous equations for a and β as shown below. Substituting (7.3) and (7.28) into (7.27) yields

$$u = a\cos\left(\frac{1}{2}\Omega t + \beta\right) + \epsilon a\left[\frac{\cos\left(\frac{3}{2}\Omega t + \beta\right)}{\Omega(\Omega + 2\omega)} + \frac{\mu\sin\left(\frac{1}{2}\Omega t + \beta\right)}{2\omega} - \frac{\cos\left(\frac{1}{2}\Omega t - \beta\right)}{2\omega\Omega}\right] + \cdots \tag{7.29}$$

Substituting (7.3) and (7.28) into (7.26) and separating real and imaginary parts, we obtain

$$\dot{a} = -\epsilon\mu a + \frac{\epsilon}{2\omega} a \sin 2\beta - \frac{\epsilon^2\mu(\Omega + 2\omega)}{4\Omega\,\omega^2} a\cos(2\beta) \tag{7.30}$$

$$\dot{\beta} = \frac{1}{2}(2\omega - \Omega) + \frac{\epsilon}{2\omega}\cos 2\beta - \frac{\epsilon^2}{2\omega}\left[\mu^2 + \frac{1}{2\omega(\Omega + 2\omega)} - \frac{\mu(\Omega + 2\omega)}{2\omega\,\Omega}\sin(2\beta)\right] \tag{7.31}$$

7.2 Multiple-Degree-of-Freedom Systems

In this section, we consider parametrically excited, nongyroscopic, multiple-degree-of-freedom systems governed by

$$\ddot{\boldsymbol{x}} + A\boldsymbol{x} + D\dot{\boldsymbol{x}} + (2\epsilon\cos\Omega t)\,F\boldsymbol{x} = 0 \tag{7.32}$$

where A, D, and F are $n \times n$ constant matrices and $\boldsymbol{x}$ is a column vector of length n. We assume that none of the eigenvalues of A is zero; this case is considered in Chapter 2. Then, we introduce the nonsingular linear transformation $\boldsymbol{x} = P\boldsymbol{u}$ into (7.32) and obtain

$$P\ddot{\boldsymbol{u}} + AP\boldsymbol{u} + DP\dot{\boldsymbol{u}} + (2\epsilon\cos\Omega t)\,FP\boldsymbol{u} = 0 \tag{7.33}$$

Multiplication of (7.33) from the left by P^{-1}, the inverse of P, yields

$$\ddot{\boldsymbol{u}} + J\boldsymbol{u} + \hat{D}\dot{\boldsymbol{u}} + (2\epsilon\cos\Omega t)\,\hat{F}\boldsymbol{u} = 0 \tag{7.34}$$

where

$$J = P^{-1}AP\,, \quad \hat{D} = P^{-1}DP\,, \quad \text{and} \quad \hat{F} = P^{-1}FP \tag{7.35}$$

One can always choose P so that J is a Jordan canonical form. In this section, we treat the case in which the eigenvalues of A are distinct and positive so that J is a diagonal matrix. We assume that $\hat{D}$ is a diagonal matrix (the so-called modal-damping assumption). Moreover, we limit our discussion to first-order expansions of the solutions of three-degree-of-freedom systems modeled by

$$\ddot{u}_1 + \omega_1^2 u_1 + 2\epsilon\mu_1\dot{u}_1 + (2\epsilon\cos\Omega t)\sum_{n=1}^{3} f_{1n}u_n = 0 \tag{7.36}$$

$$\ddot{u}_2 + \omega_2^2 u_2 + 2\epsilon\mu_2\dot{u}_2 + (2\epsilon\cos\Omega t)\sum_{n=1}^{3} f_{2n}u_n = 0 \tag{7.37}$$

$$\ddot{u}_3 + \omega_3^2 u_3 + 2\epsilon\mu_3\dot{u}_3 + (2\epsilon\cos\Omega t)\sum_{n=1}^{3} f_{3n}u_n = 0 \tag{7.38}$$

where the ω_m have been arranged so that $\omega_3 > \omega_2 > \omega_1$.

As a first step in determining uniform expansions of the solutions of (7.36)–(7.38), we recast them into a system of three first-order complex-valued equations. To accomplish this, we note that when $\epsilon = 0$ the solutions of (7.36)–(7.38) can be expressed as

$$u_n = A_n e^{i\omega_n t} + \bar{A}_n e^{-i\omega_n t} \tag{7.39a}$$

where the A_n are complex. Hence,

$$\dot{u}_n = i\omega_n \left(A_n e^{i\omega_n t} - \bar{A}_n e^{-i\omega_n t}\right) \tag{7.39b}$$

When $\epsilon \neq 0$, we continue to represent u_n and $\dot{u}_n$ as in (7.39a) and (7.39b) but with time-varying rather than constant A_n. Then, we identify $A_n e^{i\omega_n t}$ with ζ_n and rewrite (7.39a) and (7.39b) as

$$u_m = \zeta_m + \bar{\zeta}_m\,, \quad \dot{u}_m = i\omega_m \left(\zeta_m - \bar{\zeta}_m\right) \tag{7.39c}$$

Moreover, we let

$$z = e^{i\Omega t} \quad \text{and} \quad \dot{z} = i\Omega z \tag{7.40}$$

With this transformation, (7.36)–(7.38) become

$$\dot{\zeta}_1 = i\omega_1\zeta_1 - \epsilon\mu_1\left(\zeta_1 - \bar{\zeta}_1\right) + \frac{i\epsilon}{2\omega_1}(z+\bar{z})\sum_{n=1}^{3} f_{1n}\left(\zeta_n + \bar{\zeta}_n\right) \tag{7.41}$$

$$\dot{\zeta}_2 = i\omega_2\zeta_2 - \epsilon\mu_2\left(\zeta_2 - \bar{\zeta}_2\right) + \frac{i\epsilon}{2\omega_2}(z+\bar{z})\sum_{n=1}^{3} f_{2n}\left(\zeta_n + \bar{\zeta}_n\right) \tag{7.42}$$

$$\dot{\zeta}_3 = i\omega_3\zeta_3 - \epsilon\mu_3\left(\zeta_3 - \bar{\zeta}_3\right) + \frac{i\epsilon}{2\omega_3}(z+\bar{z})\sum_{n=1}^{3} f_{3n}\left(\zeta_n + \bar{\zeta}_n\right) \tag{7.43}$$

To simplify (7.41)–(7.43), we introduce the near-identity transformation

$$\zeta_m = \eta_m + \epsilon h_m\left(\eta_n, \bar{\eta}_n, z, \bar{z}\right) \tag{7.44}$$

and obtain

$$\begin{aligned}\dot{\eta}_m = {} & i\omega_m\eta_m + i\epsilon\omega_m h_m - \epsilon\mu_m\left(\eta_m - \bar{\eta}_m\right) - \epsilon\sum_{n=1}^{3}\left(\frac{\partial h_m}{\partial\eta_n}\dot{\eta}_n + \frac{\partial h_m}{\partial\bar{\eta}_n}\dot{\bar{\eta}}_n\right) \\ & - \epsilon\frac{\partial h_m}{\partial z}\dot{z} - \epsilon\frac{\partial h_m}{\partial\bar{z}}\dot{\bar{z}} + \frac{i\epsilon}{2\omega_m}(z+\bar{z})\left[f_{m1}\left(\eta_1+\bar{\eta}_1\right) + f_{m2}\left(\eta_2+\bar{\eta}_2\right)\right. \\ & \left. + f_{m3}\left(\eta_3+\bar{\eta}_3\right)\right] + \cdots\end{aligned} \tag{7.45}$$

for $m = 1, 2$, and 3. The form of the $O(\epsilon)$ terms suggests the following form for the h_m:

$$\begin{aligned}h_m = {} & \varDelta_{m1}\eta_m + \varDelta_{m2}\bar{\eta}_m + \varGamma_{m1}z\eta_1 + \varGamma_{m2}z\bar{\eta}_1 + \varGamma_{m3}z\eta_2 \\ & + \varGamma_{m4}z\bar{\eta}_2 + \varGamma_{m5}z\eta_3 + \varGamma_{m6}z\bar{\eta}_3 + \varGamma_{m7}\bar{z}\eta_1 + \varGamma_{m8}\bar{z}\bar{\eta}_1 \\ & + \varGamma_{m9}\bar{z}\eta_2 + \varGamma_{m10}\bar{z}\bar{\eta}_2 + \varGamma_{m11}\bar{z}\eta_3 + \varGamma_{m12}\bar{z}\bar{\eta}_3\end{aligned} \tag{7.46}$$

It follows from (7.45) that

$$\dot{\eta}_m = i\omega_m \eta_m + O(\epsilon) \tag{7.47}$$

Hence, substituting (7.40), (7.46), and (7.47) into the right-hand side of (7.45), we obtain

$$\begin{aligned}
\dot{\eta}_m = {} & i\omega_m \eta_m - \epsilon\mu_m \eta_m + \epsilon\left(2i\omega_m \Delta_{m2} + \mu_m\right)\bar{\eta}_m \\
& - i\epsilon\left[(\Omega + \omega_1 - \omega_m)\,\Gamma_{m1} - \frac{f_{m1}}{2\omega_m}\right] z\eta_1 \\
& - i\epsilon\left[(\Omega - \omega_1 - \omega_m)\,\Gamma_{m2} - \frac{f_{m1}}{2\omega_m}\right] z\bar{\eta}_1 \\
& - i\epsilon\left[(\Omega + \omega_2 - \omega_m)\,\Gamma_{m3} - \frac{f_{m2}}{2\omega_m}\right] z\eta_2 \\
& - i\epsilon\left[(\Omega - \omega_2 - \omega_m)\,\Gamma_{m4} - \frac{f_{m2}}{2\omega_m}\right] z\bar{\eta}_2 \\
& - i\epsilon\left[(\Omega + \omega_3 - \omega_m)\,\Gamma_{m5} - \frac{f_{m3}}{2\omega_m}\right] z\eta_3 \\
& - i\epsilon\left[(\Omega - \omega_3 - \omega_m)\,\Gamma_{m6} - \frac{f_{m3}}{2\omega_m}\right] z\bar{\eta}_3 \\
& + i\epsilon\left[(\Omega - \omega_1 + \omega_m)\,\Gamma_{m7} - \frac{f_{m1}}{2\omega_m}\right] \bar{z}\eta_1 \\
& + i\epsilon\left[(\Omega + \omega_1 + \omega_m)\,\Gamma_{m8} + \frac{f_{m1}}{2\omega_m}\right] \bar{z}\bar{\eta}_1 \\
& + i\epsilon\left[(\Omega - \omega_2 + \omega_m)\,\Gamma_{m9} + \frac{f_{m2}}{2\omega_m}\right] \bar{z}\eta_2 \\
& + i\epsilon\left[(\Omega + \omega_2 + \omega_m)\,\Gamma_{m10} + \frac{f_{m2}}{2\omega_m}\right] \bar{z}\bar{\eta}_2 \\
& + i\epsilon\left[(\Omega - \omega_3 + \omega_m)\,\Gamma_{m11} + \frac{f_{m3}}{2\omega_m}\right] \bar{z}\eta_2 \\
& + i\epsilon\left[(\Omega + \omega_3 + \omega_m)\,\Gamma_{m12} + \frac{f_{m3}}{2\omega_m}\right] \bar{z}\bar{\eta}_3 \\
& + \cdots
\end{aligned} \tag{7.48}$$

for $m = 1, 2$, and 3. Substituting (7.44) into (7.39c), we obtain to the first approximation

$$u_m = \eta_m + \bar{\eta}_m + O(\epsilon) \tag{7.49}$$

We note that (7.48) does not depend on the Δ_{m1}, and hence they are arbitrary. Choosing the Γ_{mn} to eliminate the terms involving $z\eta_m$, $z\bar{\eta}_m$, $\bar{z}\eta_m$, and $\bar{z}\bar{\eta}_m$, we find that some of the Γ_{mn} have small-divisor terms when

$$\begin{aligned}
&\Omega \approx \omega_m + \omega_n \quad \text{for} \quad m, n = 1, 2, \text{ and } 3 \\
&\Omega \approx \omega_m - \omega_n \quad \text{for} \quad m, n = 1, 2, \text{ and } 3 \quad \text{but} \quad m \neq n
\end{aligned}$$

The case $\Omega \approx 2\omega_m$, $m = 1, 2$, or 3, is called principal parametric resonance of the mth mode, which is treated in the preceding section. The case $\Omega \approx \omega_m + \omega_n$, for $m \neq n$, is called combination parametric resonance of the additive type and the case $\Omega \approx \omega_m - \omega_n$, for $m \neq n$, is called combination parametric resonance of the difference type. The cases (a) $\Omega \approx 2\omega_s$ and $\Omega \approx \omega_n \pm \omega_m$ and (b) $\Omega \approx \omega_s$ and $\Omega \approx \omega_n \pm \omega_m$ are called simultaneous parametric resonances. Next, we treat some of these cases.

7.2.1 The Case of Ω Near $\omega_2 + \omega_1$

When $\Omega \approx \omega_2 + \omega_1$ and there are no other resonances, the term involving $z\bar{\eta}_2$ is near-resonance when $m = 1$, while the term involving $z\bar{\eta}_1$ is near-resonance when $m = 2$, and there are no near-resonance terms when $m = 3$. Consequently, choosing the Δ_{m2} and Γ_{mn} to eliminate all nonresonance terms in (7.48), we obtain

$$\dot{\eta}_1 = i\omega_1\eta_1 - \epsilon\mu_1\eta_1 + \frac{i\epsilon f_{12}}{2\omega_1} z\bar{\eta}_2 + O\left(\epsilon^2\right) \tag{7.50}$$

$$\dot{\eta}_2 = i\omega_2\eta_2 - \epsilon\mu_2\eta_2 + \frac{i\epsilon f_{21}}{2\omega_2} z\bar{\eta}_1 + O\left(\epsilon^2\right) \tag{7.51}$$

$$\dot{\eta}_3 = i\omega_3\eta_3 - \epsilon\mu_3\eta_3 + O\left(\epsilon^2\right) \tag{7.52}$$

We note that Γ_{14} and Γ_{22} are arbitrary. Moreover, (7.50)–(7.52) show that, although the first and second modes are coupled, the third mode is uncoupled from the first two modes. The polar form of (7.50)–(7.52) is in full agreement with that obtained by using the method of multiple scales (Nayfeh and Mook, 1979).

7.2.2 The Case of Ω Near $\omega_2 - \omega_1$

In this case, $\eta_2\bar{z}$ and $\eta_1 z$ are near-resonance terms when $m = 1$ and 2, respectively, and there are no resonance terms when $m = 3$. Consequently, to $O(\epsilon)$, the normal form of (7.48) is

$$\dot{\eta}_1 = i\omega_1\eta_1 - \epsilon\mu_1\eta_1 + \frac{i\epsilon f_{12}}{2\omega_1} \bar{z}\eta_2 \tag{7.53}$$

$$\dot{\eta}_2 = i\omega_2\eta_2 - \epsilon\mu_2\eta_1 + \frac{i\epsilon f_{21}}{2\omega_2} z\eta_1 \tag{7.54}$$

$$\dot{\eta}_3 = i\omega_3\eta_3 - \epsilon\mu_3\eta_3 \tag{7.55}$$

7.2.3 The Case of Ω Near $\omega_2 + \omega_1$ and $\omega_3 - \omega_2$

In this case, $z\bar{\eta}_2$ is a near-resonance term when $m = 1$, $z\bar{\eta}_1$ and $\eta_3\bar{z}$ are near-resonance terms when $m = 2$, and $z\eta_2$ is a near-resonance term when $m = 3$.

Consequently, to $O(\epsilon)$, the normal form of (7.48) is

$$\dot{\eta}_1 = i\omega_1\eta_1 - \epsilon\mu_1\eta_1 + \frac{i\epsilon f_{12}}{2\omega_1}z\bar{\eta}_2 \tag{7.56}$$

$$\dot{\eta}_2 = i\omega_2\eta_2 - \epsilon\mu_2\eta_2 + \frac{i\epsilon f_{21}}{2\omega_2}z\bar{\eta}_1 + \frac{i\epsilon f_{23}}{2\omega_2}\bar{z}\eta_3 \tag{7.57}$$

$$\dot{\eta}_3 = i\omega_3\eta_3 - \epsilon\mu_3\eta_3 + \frac{i\epsilon f_{32}}{2\omega_3}z\eta_2 \tag{7.58}$$

7.2.4
The Case of Ω Near $2\omega_3$ and $\omega_2 + \omega_1$

In this case, the near-resonance terms are $z\bar{\eta}_2$ when $m = 1$, $z\bar{\eta}_1$ when $m = 2$, and $z\bar{\eta}_3$ when $m = 3$. Consequently, the normal form of (7.48) is

$$\dot{\eta}_1 = i\omega_1\eta_1 - \epsilon\mu_1\eta_1 + \frac{i\epsilon f_{12}}{2\omega_1}z\bar{\eta}_2 \tag{7.59}$$

$$\dot{\eta}_2 = i\omega_2\eta_2 - \epsilon\mu_2\eta_2 + \frac{i\epsilon f_{21}}{2\omega_2}z\bar{\eta}_1 \tag{7.60}$$

$$\dot{\eta}_3 = i\omega_3\eta_3 - \epsilon\mu_3\eta_3 + \frac{i\epsilon f_{33}}{2\omega_3}z\bar{\eta}_3 \tag{7.61}$$

We note that η_3 is uncoupled from η_1 and η_2.

7.3
Linear Systems Having Repeated Frequencies

In contrast with the preceding section, here we consider a three-degree-of-freedom system having two repeated frequencies and the Jordan form

$$J = \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 1 & \omega_1^2 & 0 \\ 0 & 0 & \omega_3^2 \end{bmatrix} \tag{7.62}$$

Thus, we consider

$$\ddot{u}_1 + \omega_1^2 u_1 + 2\epsilon\mu_1\dot{u}_1 + (2\epsilon\cos\Omega t)\sum_{n=1}^{3} f_{1n}u_n = 0 \tag{7.63}$$

$$\ddot{u}_2 + \omega_1^2 u_2 + u_1 + 2\epsilon\mu_2\dot{u}_2 + (2\epsilon\cos\Omega t)\sum_{n=1}^{3} f_{2n}u_n = 0 \tag{7.64}$$

$$\ddot{u}_3 + \omega_3^2 u_3 + 2\epsilon\mu_3\dot{u}_3 + (2\epsilon\cos\Omega t)\sum_{n=1}^{3} f_{3n}u_n = 0 \tag{7.65}$$

We assume that $\omega_3 > \omega_1$.

In the absence of damping and the parametric excitation, the system is unstable (the system is said to be in flutter). To show this, we note that when $\mu_n = 0$ and $f_{mn} = 0$,

$$u_1 = a_1 \cos(\omega_1 t + \beta_1)$$
$$u_3 = a_3 \cos(\omega_3 t + \beta_3)$$

where a_1, a_3, β_1 and β_3 are constants. Then (7.64) becomes

$$\ddot{u}_2 + \omega_1^2 u_2 = -u_1 = -a_1 \cos(\omega_1 t + \beta_1)$$

Hence,

$$u_2 = a_2 \cos(\omega_2 t + \beta_2) - \frac{a_1}{2\omega_1} t \sin(\omega_1 t + \beta_1)$$

which contains a secular term or resonance term. Consequently, one refers to this one-to-one resonance as a *nonsemisimple one-to-one resonance* and the linear operator J is said to have a *generic nonsemisimple structure*.

We wish to determine the normal forms of (7.63)–(7.65) in the presence of damping and the parametric excitation so that one can use these forms to ascertain if the damping and parametric excitation can stabilize the system. In the stable case, one assumes that all of the three variables u_1, u_2, and u_3 are bounded and, if possible, determine the values of the parameters which are consistent with this assumption.

Although the damping and parametric excitation might stabilize the system, we still expect the amplitude of u_2 to be much larger than those of u_1 and u_3. We use this observation to simplify the obtained normal forms.

As a first step in the application of the method of normal forms, we recast (7.63)–(7.65) in complex-valued form by using the transformation (7.39c) and (7.40). The result is

$$\dot{\zeta}_1 = i\omega_1\zeta_1 - \epsilon\mu_1(\zeta_1 - \bar{\zeta}_1) + \frac{i\epsilon}{2\omega_1}(z + \bar{z})\sum_{n=1}^{3} f_{1n}(\zeta_n + \bar{\zeta}_n) \tag{7.66}$$

$$\dot{\zeta}_2 = i\omega_1\zeta_2 + \frac{i}{2\omega_1}(\zeta_1 + \bar{\zeta}_1) - \epsilon\mu_2(\zeta_2 - \bar{\zeta}_2) + \frac{i\epsilon}{2\omega_1}(z + \bar{z})\sum_{n=1}^{3} f_{2n}(\zeta_n + \bar{\zeta}_n) \tag{7.67}$$

$$\dot{\zeta}_3 = i\omega_3\zeta_3 - \epsilon\mu_3(\zeta_3 - \bar{\zeta}_3) + \frac{i\epsilon}{2\omega_3}(z + \bar{z})\sum_{n=1}^{3} f_{3n}(\zeta_n + \bar{\zeta}_n) \tag{7.68}$$

First, we introduce a transformation to simplify the first-order problem. We note that $(i\zeta_1)/(2\omega_1)$ is a resonance term in (7.67) and hence it cannot be eliminated by any transformation. This leaves the term $(i\bar{\zeta}_1)/(2\omega_1)$, which can be eliminated by the transformation

$$\zeta_1 = \eta_1\,, \quad \zeta_2 = \eta_2 + \Delta_1\bar{\eta}_1\,, \quad \zeta_3 = \eta_3 \tag{7.69}$$

Substituting (7.69) into the $O(1)$ terms in (7.66) and (7.67) yields

$$\dot{\eta}_1 = i\omega_1\eta_1 + O(\epsilon) \tag{7.70}$$

$$\dot{\eta}_2 = i\omega_1\eta_2 + i\omega_1\Delta_1\bar{\eta}_1 - \Delta_1\dot{\bar{\eta}}_1 + \frac{i}{2\omega_1}(\eta_1 + \bar{\eta}_1) + O(\epsilon) \tag{7.71}$$

Using (7.70) in (7.71), we have

$$\dot{\eta}_2 = i\omega_1\eta_2 + \frac{i}{2\omega_1}\eta_1 + \left(2i\omega_1\Delta_1 + \frac{i}{2\omega_1}\right)\bar{\eta}_1 + O(\epsilon) \tag{7.72}$$

Then, choosing Δ_1 to eliminate $\bar{\eta}_1$ from (7.72) yields

$$\Delta_1 = -\frac{1}{4\omega_1^2} \tag{7.73}$$

Substituting for ζ_2 from (7.69) into (7.39c) and using (7.73), we have

$$u_2 = \eta_2 + \bar{\eta}_2 - \frac{1}{4\omega_1^2}(\eta_1 + \bar{\eta}_1) + O(\epsilon) \tag{7.74}$$

Substituting (7.69) into (7.66)–(7.68) yields

$$\dot{\eta}_1 = i\omega_1\eta_1 - \epsilon\mu_1(\eta_1 - \bar{\eta}_1) + \frac{i\epsilon}{2\omega_1}(z + \bar{z})\sum_{n=1}^{3} f_{1n}(\eta_n + \bar{\eta}_n) - \frac{i\epsilon}{8\omega_1^3} f_{12}(z + \bar{z})(\eta_1 + \bar{\eta}_1) \tag{7.75}$$

$$\dot{\eta}_2 = i\omega_1\eta_2 + \frac{i}{2\omega_1}\eta_1 - \epsilon\mu_2(\eta_2 - \bar{\eta}_2) + \frac{i\epsilon}{2\omega_1}(z + \bar{z})\sum_{n=1}^{3} f_{2n}(\eta_n + \bar{\eta}_n) - \frac{i\epsilon}{8\omega_1^3} f_{22}(z + \bar{z})(\eta_1 + \bar{\eta}_1) \tag{7.76}$$

$$\dot{\eta}_3 = i\omega_3\eta_3 - \epsilon\mu_3(\eta_3 - \bar{\eta}_3) + \frac{i\epsilon}{2\omega_3}(z + \bar{z})\sum_{n=1}^{3} f_{3n}(\eta_n + \bar{\eta}_n) - \frac{i\epsilon}{8\omega_1^2\omega_3} f_{32}(z + \bar{z})(\eta_1 + \bar{\eta}_1) \tag{7.77}$$

To simplify (7.75)–(7.77), we introduce the near-identity transformation

$$\eta_m = \xi_m + \epsilon h_m\left(\xi_n, \bar{\xi}_n, z, \bar{z}\right) \tag{7.78}$$

and choose the h_m to eliminate the nonresonance terms. The result depends on the resonance conditions. Three of these conditions, namely $\Omega \approx 2\omega_1$, $\Omega \approx \omega_3 + \omega_1$, and $\Omega \approx \omega_3 - \omega_1$, are discussed next. In Section 7.3.4, we carry out the transformation to second order for the case $\Omega \approx \omega_1$.

7.3.1
The Case of Ω Near $2\omega_1$

When $\Omega \approx 2\omega_1$ and there are no other resonances, the terms $z\bar{\eta}_1$ and $z\bar{\eta}_2$ are near-resonance terms in both (7.75) and (7.76). Hence, the normal form of (7.75)–(7.77) is

$$\dot{\xi}_1 = i\omega_1\xi_1 - \epsilon\mu_1\xi_1 + \frac{i\epsilon}{2\omega_1}\left(f_{11}z\bar{\xi}_1 + f_{12}z\bar{\xi}_2\right) - \frac{i\epsilon}{8\omega_1^3} f_{12}z\bar{\xi}_1 \tag{7.79}$$

$$\dot{\xi}_2 = i\omega_1\xi_2 + \frac{i}{2\omega_1}\xi_1 - \epsilon\mu_2\xi_2 + \frac{i\epsilon}{2\omega_1}\left(f_{21}z\bar{\xi}_1 + f_{22}z\bar{\xi}_2\right) - \frac{i\epsilon}{8\omega_1^3} f_{22}z\bar{\xi}_1 \tag{7.80}$$

$$\dot{\xi}_3 = i\omega_3\xi_3 - \epsilon\mu_3\xi_3 \tag{7.81}$$

As noted earlier, (7.79) and (7.80) can be further simplified because u_2 is much larger than u_1 and hence ξ_2 is much larger than ξ_1. To accomplish this simplification, we scale the damping coefficients μ_1 and μ_2 and the variables ξ_1 and ξ_2 as

$$\xi_1 = \chi_1\,, \quad \xi_2 = \epsilon^{-\lambda_2}\chi_2\,, \quad \mu_n = \epsilon^{-\lambda_1}\hat{\mu}_n$$

where χ_1 and χ_2 are $O(1)$ and λ_1 and λ_2 are positive constants to be determined from the analysis. Because (7.79) and (7.80) are linear and homogeneous, we ordered ξ_1 at $O(\epsilon)$ without loss of generality. Using these scalings, we rewrite (7.79) and (7.80) as

$$\dot{\chi}_1 = i\omega_1\chi_1 - \epsilon^{1-\lambda_1}\hat{\mu}_1\chi_1 + \frac{i}{2\omega_1}\left(\epsilon f_{11}z\bar{\chi}_1 + \epsilon^{1-\lambda_2} f_{12}z\bar{\chi}_2\right) - \frac{i}{8\omega_1^3}\epsilon f_{12}z\bar{\chi}_1 \tag{7.82}$$

$$\dot{\chi}_2 = i\omega_1\chi_2 + \frac{i}{2\omega_1}\epsilon^{\lambda_2}\chi_1 - \epsilon^{1-\lambda_1}\hat{\mu}_2\chi_2 + \frac{i}{2\omega_1}\left(\epsilon^{1+\lambda_2} f_{21}z\bar{\chi}_1 + \epsilon f_{22}z\bar{\chi}_2\right) - \frac{i}{8\omega_1^3}\epsilon^{1+\lambda_2} f_{22}z\bar{\chi}_1 \tag{7.83}$$

As $\epsilon \to 0$, we note that $\epsilon f_{11}z\hat{\chi}_1$ is small compared with $\epsilon^{1-\lambda_2} f_{12}z\bar{\chi}_2$ because $\lambda_2 > 0$. Then, requiring the damping term in (7.82) be the same order as the resonance term $i(2\omega_1)^{-1}\epsilon^{1-\lambda_2} f_{12}z\bar{\chi}_2$, we have

$$1 - \lambda_1 = 1 - \lambda_2 \quad \text{or} \quad \lambda_1 = \lambda_2 \tag{7.84}$$

Similarly, as $\epsilon \to 0$, the resonance terms (the terms inside the parenthesis) in (7.83) are small compared with the damping term $\epsilon^{1-\lambda_1}\hat{\mu}_2\chi_2$ because λ_1 and λ_2 are positive. Then, requiring $i(2\omega_1)^{-1}\epsilon^{\lambda_2}\chi_1$ to be the same order as $\epsilon^{1-\lambda_1}\hat{\mu}_2\chi_2$, we have

$$\lambda_2 = 1 - \lambda_1 \tag{7.85}$$

Hence, $\lambda_1 = \lambda_2 = 1/2$. Consequently, keeping terms up to $O(\epsilon^{1/2})$ in (7.82) and (7.83), we obtain the simplified normal form

$$\dot{\chi}_1 = i\omega_1\chi_1 - \epsilon^{1/2}\hat{\mu}_1\chi_1 + \frac{i\epsilon^{1/2}}{2\omega_1} f_{12} z\bar{\chi}_2 + \cdots \tag{7.86}$$

$$\dot{\chi}_2 = i\omega_2\chi_2 - \epsilon^{1/2}\hat{\mu}_2\chi_2 + \frac{i\epsilon^{1/2}}{2\omega_1}\chi_1 + \cdots \tag{7.87}$$

Equations 7.86 and 7.87 are also called the distinguished limit or least degenerate form of (7.82) and (7.83). We note that (7.86) and (7.87) agree with those obtained by Nayfeh and Mook (1979) by using the method of multiple scales.

7.3.2 The Case of Ω Near $\omega_3 + \omega_1$

In this case, the term $z\bar{\eta}_3$ is near-resonance in (7.75) and (7.76), while the terms $z\bar{\eta}_1$ and $z\bar{\eta}_2$ are near-resonance in (7.77). Hence, the normal form of (7.75)–(7.77) is

$$\dot{\xi}_1 = i\omega_1\xi_1 - \epsilon\mu_1\xi_1 + \frac{i\epsilon}{2\omega_1} f_{13} z\bar{\xi}_3 \tag{7.88}$$

$$\dot{\xi}_2 = i\omega_1\xi_2 + \frac{i}{2\omega_1}\xi_1 - \epsilon\mu_2\xi_2 + \frac{i\epsilon}{2\omega_1} f_{23} z\bar{\xi}_3 \tag{7.89}$$

$$\dot{\xi}_3 = i\omega_3\xi_3 - \epsilon\mu_3\xi_3 + \frac{i\epsilon}{2\omega_3}\left(f_{31} z\bar{\xi}_1 + f_{32} z\bar{\xi}_2\right) - \frac{i\epsilon}{8\omega_1^2\omega_3} f_{32} z\bar{\xi}_1 \tag{7.90}$$

Again, using the fact that ξ_2 is much larger than ξ_1 and ξ_3, we further simplify (7.88)–(7.90). To accomplish this, we scale the ξ_n and μ_n as

$$\xi_1 = \chi_1\,, \quad \xi_2 = \epsilon^{-\lambda_2}\chi_2\,, \quad \xi_3 = \epsilon^{-\lambda_3}\chi_3\,, \quad \mu_n = \epsilon^{-\lambda_1}\hat{\mu}_n \tag{7.91}$$

where the χ_n and $\hat{\mu}_n$ are $O(1)$ and the λ_n are positive constants. Substituting (7.91) into (7.88)–(7.90), we obtain

$$\dot{\chi}_1 = i\omega_1\chi_1 - \epsilon^{1-\lambda_1}\hat{\mu}_1\chi_1 + \frac{i}{2\omega_1}\epsilon^{1-\lambda_3} f_{13} z\bar{\chi}_3 \tag{7.92}$$

$$\dot{\chi}_2 = i\omega_1\chi_2 + \frac{i}{2\omega_1}\epsilon^{\lambda_2}\chi_1 - \epsilon^{1-\lambda_1}\hat{\mu}_2\chi_2 + \frac{i}{2\omega_1}\epsilon^{1+\lambda_2-\lambda_3} f_{23} z\bar{\chi}_3 \tag{7.93}$$

$$\begin{aligned}\chi_3 = {}& i\omega_3\chi_3 - \epsilon^{1-\lambda_1}\hat{\mu}_3\chi_3 + \frac{i}{2\omega_3}\epsilon^{1+\lambda_3} f_{31} z\bar{\chi}_1 + \frac{i}{2\omega_3}\epsilon^{1+\lambda_3-\lambda_2} f_{32} z\bar{\chi}_2 \\ & - \frac{i}{8\omega_1^2\omega_3}\epsilon^{1+\lambda_3} f_{32} z\bar{\chi}_1\end{aligned} \tag{7.94}$$

As $\epsilon \to 0$, the distinguished limit of (7.92)–(7.94) corresponds to

$$1 - \lambda_1 = 1 - \lambda_3$$
$$1 - \lambda_1 = \lambda_2$$
$$1 - \lambda_1 = 1 + \lambda_3 - \lambda_2$$

or

$$\lambda_2 = \tfrac{2}{3} \quad \text{and} \quad \lambda_1 = \lambda_3 = \tfrac{1}{3}$$

Thus, to the first approximation, (7.92)–(7.94) simplify to the normal form

$$\dot{\chi}_1 = i\omega_1\chi_1 - \epsilon^{2/3}\hat{\mu}_1\chi_1 + \frac{i\epsilon^{2/3}}{2\omega_1} f_{13} z\bar{\chi}_3 + \cdots \tag{7.95}$$

$$\dot{\chi}_2 = i\omega_1\chi_2 - \epsilon^{2/3}\hat{\mu}_2\chi_2 + \frac{i\epsilon^{2/3}}{2\omega_1}\chi_1 + \cdots \tag{7.96}$$

$$\dot{\chi}_3 = i\omega_3\chi_3 - \epsilon^{2/3}\hat{\mu}_3\chi_3 + \frac{i}{2\omega_3}\epsilon^{2/3} f_{32} z\bar{\chi}_2 + \cdots \tag{7.97}$$

Equations 7.95–7.97 agree with those obtained by Nayfeh and Mook (1979) by using the method of multiple scales.

7.3.3
The Case of Ω Near $\omega_3 - \omega_1$

In this case, the term $\bar{z}\xi_3$ is near-resonance in (7.75) and (7.76), while the terms $z\xi_1$ and $z\xi_2$ are near-resonance in (7.77). Hence, the normal form of (7.75)–(7.77) is

$$\dot{\xi}_1 = i\omega_1\xi_1 - \epsilon\mu_1\xi_1 + \frac{i\epsilon}{2\omega_1} f_{13}\bar{z}\xi_3 \tag{7.98}$$

$$\dot{\xi}_2 = i\omega_1\xi_2 + \frac{i}{2\omega_1}\xi_1 - \epsilon\mu_2\xi_2 + \frac{i\epsilon}{2\omega_1} f_{23}\bar{z}\xi_3 \tag{7.99}$$

$$\dot{\xi}_3 = i\omega_3\xi_3 - \epsilon\mu_3\xi_3 + \frac{i\epsilon}{2\omega_3}\left(f_{31}z\xi_1 + f_{32}z\xi_2\right) - \frac{i\epsilon}{8\omega_1^2\omega_3} f_{32} z\xi_1 \tag{7.100}$$

As in Section 7.3.2, (7.98)–(7.100) can be further simplified by using the scaling

$$\xi_1 = \chi_1\,, \quad \xi_2 = \epsilon^{-2/3}\chi_2\,, \quad \xi_3 = \epsilon^{-1/3}\chi_3\,, \quad \mu_n = \epsilon^{-1/3}\hat{\mu}_n$$

Substituting these scaled expressions into (7.98)–(7.100) and keeping the leading terms in ϵ (i.e., up to $O(\epsilon^{2/3})$), we obtain the simplified normal form

$$\dot{\chi}_1 = i\omega_1\chi_1 - \epsilon^{2/3}\hat{\mu}_1\chi_1 + \frac{i\epsilon^{2/3}}{2\omega_1} f_{13}\hat{z}\chi_3 + \cdots \tag{7.101}$$

$$\dot{\chi}_2 = i\omega_1\chi_2 - \epsilon^{2/3}\hat{\mu}_2\chi_2 + \frac{i\epsilon^{2/3}}{2\omega_1}\chi_1 + \cdots \tag{7.102}$$

$$\dot{\chi}_3 = i\omega_3\chi_3 - \epsilon^{2/3}\hat{\mu}_3\chi_3 + \frac{i\epsilon^{2/3}}{2\omega_3} f_{32} z\chi_2 + \cdots \tag{7.103}$$

7.3.4
The Case of Ω Near ω_1

In this case, instead of first determining the normal form of (7.66)–(7.68) and then scaling the dependent variables to simplify the obtained normal forms, we first scale the dependent variables and then obtain the simplified normal form, thereby reducing the algebra involved. Because ω_3 is away from ω_1 and hence $\Omega \approx \omega_1$ is away from ω_3, we assume that ζ_3 is $O(1)$, which is the same order as ζ_1. However, because ζ_2 is large compared with ζ_1, we assume that $\zeta_2 = O(\epsilon^{-\lambda_2})$, where λ_2 is positive. Therefore, we let

$$\zeta_1 = \eta_1 \,, \quad \zeta_2 = \epsilon^{-\lambda_2}\eta_2 \,, \quad \text{and} \quad \zeta_3 = \eta_3 \tag{7.104}$$

and rewrite (7.66)–(7.68) as

$$\dot{\eta}_1 = i\omega_1\eta_1 - \epsilon\mu_1(\eta_1 - \bar{\eta}_1) + \frac{i}{2\omega_1}(z+\bar{z}) \times \left[\epsilon f_{11}(\eta_1+\bar{\eta}_1) + \epsilon^{1-\lambda_2} f_{12}(\eta_2+\bar{\eta}_2) + \epsilon f_{13}(\eta_3+\bar{\eta}_3)\right] \tag{7.105}$$

$$\dot{\eta}_2 = i\omega_1\eta_2 + \frac{i}{2\omega_1}\epsilon^{\lambda_2}(\eta_1+\bar{\eta}_1) - \epsilon\mu_2(\eta_2-\bar{\eta}_2) + \frac{i}{2\omega_1}(z+\bar{z}) \bullet \left[\epsilon^{1+\lambda_2} f_{21}(\eta_1+\bar{\eta}_1) + \epsilon f_{22}(\eta_2+\bar{\eta}_2) + \epsilon^{1+\lambda_2} f_{23}(\eta_3+\bar{\eta}_3)\right] \tag{7.106}$$

$$\dot{\eta}_3 = i\omega_3\eta_3 - \epsilon\mu_3(\eta_3 - \bar{\eta}_3) + \frac{i}{2\omega_3}(z+\bar{z}) \times \left[\epsilon f_{31}(\eta_1+\bar{\eta}_1) + \epsilon^{1-\lambda_2} f_{32}(\eta_2+\bar{\eta}_2) + \epsilon f_{33}(\eta_3+\bar{\eta}_3)\right] \tag{7.107}$$

Because $\Omega \approx \omega_1$ and is away from ω_3, the term $(z+\bar{z})(\eta_2+\bar{\eta}_2)$ is a nonresonance term and hence we choose $\lambda_2 = 1$, making the terms $i(2\omega_1)^{-1}(z+\bar{z}) f_{12}(\eta_2+\bar{\eta}_2)$ and $i(2\omega_3)^{-1}(z+\bar{z}) f_{32}(\eta_2+\bar{\eta}_2)$ in (7.105) and (7.107), respectively, of $O(1)$. Hence, to simplify (7.105)–(7.107), we let

$$\eta_1 = \xi_1 + h_{10}(z,\bar{z},\xi_2,\bar{\xi}_2) + \epsilon h_{11}(z,\bar{z},\xi_n,\bar{\xi}_n) + \cdots \tag{7.108}$$

$$\eta_2 = \xi_2 + \epsilon h_{21}(z,\bar{z},\xi_n,\bar{\xi}_n) + \cdots \tag{7.109}$$

$$\eta_3 = \xi_3 + h_{30}(z,\bar{z},\xi_2,\bar{\xi}_2) + \epsilon h_{31}(z,\bar{z},\xi_n,\bar{\xi}_n) + \cdots \tag{7.110}$$

where

$$\dot{\xi}_1 = i\omega_1\xi_1 + \epsilon g_1(z,\bar{z},\xi_n,\bar{\xi}_n) + \cdots \tag{7.111}$$

$$\dot{\xi}_2 = i\omega_1\xi_2 + \epsilon g_2(z,\bar{z},\xi_n,\bar{\xi}_n) + \cdots \tag{7.112}$$

$$\dot{\xi}_3 = i\omega_3\xi_3 + \epsilon g_3(z,\bar{z},\xi_n,\bar{\xi}_n) + \cdots \tag{7.113}$$

We have included the terms h_{10} and h_{30} at $O(1)$ to eliminate the terms of $O(1)$ in (7.105) and (7.107). Substituting (7.108)–(7.113) into (7.105)–(7.107) using (7.40), and equating the coefficients of ϵ^0 and ϵ on both sides, we obtain

$$i\Omega\left(\frac{\partial h_{10}}{\partial z}z-\frac{\partial h_{10}}{\partial \bar{z}}\bar{z}\right)+i\omega_1\left[\frac{\partial h_{10}}{\partial \xi_2}\xi_2-\frac{\partial h_{10}}{\partial \bar{\xi}_2}\bar{\xi}_2-h_{10}\right]$$
$$=\frac{i}{2\omega_1}f_{12}\left(z+\bar{z}\right)\left(\xi_2+\bar{\xi}_2\right) \tag{7.114}$$

$$i\Omega\left(\frac{\partial h_{30}}{\partial z}z-\frac{\partial h_{30}}{\partial \bar{z}}\bar{z}\right)+i\omega_1\left[\frac{\partial h_{30}}{\partial \xi_2}\xi_2-\frac{\partial h_{30}}{\partial \bar{\xi}_2}\bar{\xi}_2\right]-i\omega_3h_{30}$$
$$=\frac{i}{2\omega_3}f_{32}\left(z+\bar{z}\right)\left(\xi_2+\bar{\xi}_2\right) \tag{7.115}$$

$$g_1+i\Omega\left(\frac{\partial h_{11}}{\partial z}z-\frac{\partial h_{11}}{\partial \bar{z}}\bar{z}\right)+\sum_{n=1}^{3}i\omega_n\left(\frac{\partial h_{11}}{\partial \xi_n}\xi_n-\frac{\partial h_{11}}{\partial \bar{\xi}_n}\bar{\xi}_n\right)-i\omega_1h_{11}$$
$$=-\frac{\partial h_{10}}{\partial \xi_2}g_2-\frac{\partial h_{10}}{\partial \bar{\xi}_2}\bar{g}_2-\mu_1\left(\xi_1-\bar{\xi}_1+h_{10}-\bar{h}_{10}\right)$$
$$+\frac{i}{2\omega_1}f_{11}\left(z+\bar{z}\right)\left(\xi_1+\bar{\xi}_1+h_{10}+\bar{h}_{10}\right)$$
$$+\frac{i}{2\omega_1}f_{12}\left(z+\bar{z}\right)\left(h_{21}+\bar{h}_{21}\right)$$
$$+\frac{i}{2\omega_1}f_{13}\left(z+\bar{z}\right)\left(\xi_3+\bar{\xi}_3+h_{30}+\bar{h}_{30}\right) \tag{7.116}$$

$$g_2+i\Omega\left(\frac{\partial h_{21}}{\partial z}z-\frac{\partial h_{21}}{\partial \bar{z}}\bar{z}\right)+\sum_{n=1}^{3}i\omega_n\left(\frac{\partial h_{21}}{\partial \xi_n}\xi_n-\frac{\partial h_{21}}{\partial \bar{\xi}_n}\bar{\xi}_n\right)-i\omega_1h_{21}$$
$$=\frac{i}{2\omega_1}\left(\xi_1+\bar{\xi}_1+h_{10}+\bar{h}_{10}\right)-\mu_2\left(\xi_2-\bar{\xi}_2\right)+\frac{i}{2\omega_1}f_{22}\left(z+\bar{z}\right)\left(\xi_2+\bar{\xi}_2\right) \tag{7.117}$$

$$g_3+i\Omega\left(\frac{\partial h_{31}}{\partial z}z-\frac{\partial h_{31}}{\partial \bar{z}}\bar{z}\right)+\sum_{n=1}^{3}i\omega_n\left(\frac{\partial h_{31}}{\partial \xi_n}\xi_n-\frac{\partial h_{31}}{\partial \bar{\xi}_n}\bar{\xi}_n\right)-i\omega_3h_{31}$$
$$=-\frac{\partial h_{30}}{\partial \xi_2}g_2-\frac{\partial h_{30}}{\partial \bar{z}}\bar{g}_2-\mu_3\left(\xi_3-\bar{\xi}_3+h_{30}-\bar{h}_{30}\right)$$
$$+\frac{i}{2\omega_3}f_{31}\left(z+\bar{z}\right)\left(\xi_1+\bar{\xi}_1+h_{10}+\bar{h}_{10}\right)$$
$$+\frac{i}{2\omega_3}f_{32}\left(z+\bar{z}\right)\left(h_{21}+\bar{h}_{21}\right)$$
$$+\frac{i}{2\omega_3}f_{33}\left(z+\bar{z}\right)\left(\xi_3+\bar{\xi}_3+h_{30}+\bar{h}_{30}\right) \tag{7.118}$$

where $\omega_2=\omega_1$.

The form of the right-hand sides of (7.114) and (7.115) suggests the following forms for h_{10} and h_{30}:

$$h_{n0}=\Gamma_{n1}z\xi_2+\Gamma_{n2}z\bar{\xi}_2+\Gamma_{n3}\bar{z}\xi_2+\Gamma_{n4}\bar{z}\bar{\xi}_2 \tag{7.119}$$

for $n = 1$ and 3. Substituting (7.119) into (7.114) and (7.115) and equating each of the coefficients of $z\xi_2$, $z\bar{\xi}_2$, $\bar{z}\xi_2$, and $\bar{z}\bar{\xi}_2$ on both sides, we obtain

$$\Gamma_{11} = \frac{f_{12}}{2\omega_1\Omega}, \quad \Gamma_{12} = \frac{f_{12}}{2\omega_1(\Omega - 2\omega_1)},$$
$$\Gamma_{13} = -\frac{f_{12}}{2\omega_1\Omega}, \quad \Gamma_{14} = -\frac{f_{12}}{2\omega_1(\Omega + 2\omega_1)}, \tag{7.120}$$

$$\Gamma_{31} = \frac{f_{32}}{2\omega_3(\Omega + \omega_1 - \omega_3)}, \quad \Gamma_{32} = \frac{f_{32}}{2\omega_3(\Omega - \omega_1 - \omega_3)},$$
$$\Gamma_{33} = -\frac{f_{32}}{2\omega_3(\Omega - \omega_1 + \omega_3)}, \quad \Gamma_{34} = -\frac{f_{32}}{2\omega_3(\Omega + \omega_1 + \omega_3)} \tag{7.121}$$

Substituting (7.119) into (7.117) yields

$$g_2 + i\Omega\left(\frac{\partial h_{21}}{\partial z}z - \frac{\partial h_{21}}{\partial \bar{z}}\bar{z}\right) + \sum_{n=1}^{3} i\omega_n\left(\frac{\partial h_{21}}{\partial \xi_n}\xi_n - \frac{\partial h_{21}}{\partial \bar{\xi}_n}\bar{\xi}_n\right) - i\omega_1 h_{21}$$
$$= \frac{i}{2\omega_1}\left(\xi_1 + \bar{\xi}_1\right) - \mu_2\left(\xi_2 - \bar{\xi}_2\right) + \frac{i}{2\omega_1} f_{22}\left(z + \bar{z}\right)\left(\xi_2 + \bar{\xi}_2\right)$$
$$+ \frac{i}{2\omega_1}\left[\left(\Gamma_{11} + \Gamma_{14}\right)\left(z\xi_2 + \bar{z}\bar{\xi}_2\right) + \left(\Gamma_{12} + \Gamma_{13}\right)\left(z\bar{\xi}_2 + \bar{z}\xi_2\right)\right] \tag{7.122}$$

Equating g_2 to the resonance terms on the right-hand side of (7.122), we have

$$g_2 = \frac{i}{2\omega_1}\xi_1 - \mu_2\xi_2 \tag{7.123}$$

Then, the form of the remaining terms in (7.122) suggests the following form for h_{21}:

$$h_{21} = \Gamma_{21}z\xi_2 + \Gamma_{22}z\bar{\xi}_2 + \Gamma_{23}\bar{z}\xi_2 + \Gamma_{24}\bar{z}\bar{\xi}_2 + \Gamma_{25}\bar{\xi}_2 + \Gamma_{26}\bar{\xi}_1 \tag{7.124}$$

Substituting (7.124) into (7.122), using (7.123), and equating each of the coefficients of $z\xi_2$, $z\bar{\xi}_2$, $\bar{z}\xi_2$, $\bar{z}\bar{\xi}_2$, $\bar{\xi}_2$, and $\bar{\xi}_1$ on both sides, we obtain

$$\Gamma_{21} = \frac{\Gamma_{11} + \Gamma_{14} + f_{22}}{2\omega_1\Omega}, \quad \Gamma_{22} = \frac{\Gamma_{12} + \Gamma_{13} + f_{22}}{2\omega_1(\Omega - 2\omega_1)},$$
$$\Gamma_{23} = -\frac{\Gamma_{12} + \Gamma_{13} + f_{22}}{2\omega_1\Omega},$$
$$\Gamma_{24} = -\frac{\Gamma_{11} + \Gamma_{14} + f_{22}}{2\omega_1(\Omega + 2\omega_1)}, \quad \Gamma_{25} = \frac{i\mu_2}{2\omega_1}, \quad \Gamma_{26} = -\frac{1}{4\omega_1^2} \tag{7.125}$$

We note that we needed to explicitly determine h_{21} because it is needed to determine g_1 from (7.116).

Substituting (7.119) and (7.124) into (7.116) and (7.118) yields

$$
\begin{aligned}
&g_1 + i\Omega\left(\frac{\partial h_{11}}{\partial z}z - \frac{\partial h_{11}}{\partial \bar{z}}\bar{z}\right) + \sum_{n=1}^{3} i\omega_n\left(\frac{\partial h_{11}}{\partial \xi_n}\xi_n - \frac{\partial h_{11}}{\partial \bar{\xi}_n}\bar{\xi}_n\right) - i\omega_1 h_{11} \\
&= -\mu_1\left[\xi_1 - \bar{\xi}_1 + (\Gamma_{11} - \Gamma_{14})\left(z\xi_2 - \bar{z}\bar{\xi}_2\right) + (\Gamma_{12} - \Gamma_{13})\left(z\bar{\xi}_2 - \bar{z}\xi_2\right)\right] \\
&\quad - (\Gamma_{11}z + \Gamma_{13}\bar{z})\, g_2 - (\Gamma_{12}z + \Gamma_{14}\bar{z})\, \bar{g}_2 + \frac{i}{2\omega_1} f_{11}(z+\bar{z})\, \Lambda_1 \\
&\quad + \frac{i}{2\omega_1} f_{12}(z+\bar{z})\, \Lambda_2 + \frac{i}{2\omega_1} f_{13}(z+\bar{z})\, \Lambda_3
\end{aligned}
\tag{7.126}
$$

$$
\begin{aligned}
&g_3 + i\Omega\left(\frac{\partial h_{31}}{\partial z}z - \frac{\partial h_{31}}{\partial \bar{z}}\bar{z}\right) + \sum_{n=1}^{3} i\omega_n\left(\frac{\partial h_{31}}{\partial \xi_n}\xi_n - \frac{\partial h_{31}}{\partial \bar{\xi}_n}\bar{\xi}_n\right) - i\omega_3 h_{31} \\
&= -\mu_3\left[\xi_3 - \bar{\xi}_3 + (\Gamma_{31} - \Gamma_{34})\left(z\xi_2 - \bar{z}\bar{\xi}_2\right) + (\Gamma_{32} - \Gamma_{33})\left(z\bar{\xi}_2 - \bar{z}\xi_2\right)\right] \\
&\quad - (\Gamma_{31}z + \Gamma_{33}\bar{z})\, g_2 - (\Gamma_{32}z + \Gamma_{34}\bar{z})\, \bar{g}_2 + \frac{i}{2\omega_3} f_{31}(z+\bar{z})\, \Lambda_1 \\
&\quad + \frac{i}{2\omega_3} f_{32}(z+\bar{z})\, \Lambda_2 + \frac{i}{2\omega_3} f_{33}(z+\bar{z})\, \Lambda_3
\end{aligned}
\tag{7.127}
$$

where

$$
\begin{aligned}
\Lambda_1 &= \xi_1 + \bar{\xi}_1 + (\Gamma_{11} + \Gamma_{14})\left(z\xi_2 + \bar{z}\bar{\xi}_2\right) + (\Gamma_{12} + \Gamma_{13})\left(z\bar{\xi}_2 + \bar{z}\xi_2\right) \\
\Lambda_2 &= (\Gamma_{21} + \Gamma_{24})\left(z\xi_2 + \bar{z}\bar{\xi}_2\right) + (\Gamma_{22} + \Gamma_{23})\left(z\bar{\xi}_2 + \bar{z}\xi_2\right) \\
&\quad + \frac{i\mu_2}{2\omega_1}\left(\bar{\xi}_2 - \xi_2\right) - \frac{1}{4\omega_1^2}\left(\xi_1 + \bar{\xi}_1\right) \\
\Lambda_3 &= \xi_3 + \bar{\xi}_3 + (\Gamma_{31} + \Gamma_{34})\left(z\xi_2 + \bar{z}\bar{\xi}_2\right) + (\Gamma_{32} + \Gamma_{33})\left(z\bar{\xi}_2 + \bar{z}\xi_2\right)
\end{aligned}
$$

If we do not want to continue the expansion beyond the terms in (7.108)–(7.110), we do not need to explicitly calculate h_{11} and h_{31}, and all what we need is to determine g_1 and g_3 by equating them to the resonance and near-resonance terms on the right-hand sides of (7.126) and (7.127). The result is

$$
g_1 = -\mu_1\xi_1 + \frac{i}{2\omega_1}\alpha_1 z\bar{z}\xi_2 + \frac{i}{2\omega_1}\alpha_2 z^2\bar{\xi}_2 \tag{7.128}
$$

$$
g_3 = -\mu_3\xi_3 \tag{7.129}
$$

where

$$
\begin{aligned}
\alpha_1 = {} & f_{11}(\Gamma_{11} + \Gamma_{14} + \Gamma_{12} + \Gamma_{13}) + f_{12}(\Gamma_{21} + \Gamma_{24} + \Gamma_{22} + \Gamma_{23}) \\
& + f_{13}(\Gamma_{31} + \Gamma_{34} + \Gamma_{32} + \Gamma_{33})
\end{aligned}
\tag{7.130}
$$

$$
\alpha_2 = f_{11}(\Gamma_{12} + \Gamma_{13}) + f_{12}(\Gamma_{22} + \Gamma_{23}) + f_{13}(\Gamma_{32} + \Gamma_{33}) \tag{7.131}
$$

Substituting for the Γ_{mn} from (7.120), (7.121), and (7.125) into (7.130) and (7.131), we obtain

$$\alpha_1 = \frac{2f_{12}(f_{11}+f_{22})}{\Omega^2-4\omega_1^2} + \frac{f_{12}^2}{\Omega^2}\left[\frac{1}{(\Omega+2\omega_1)^2} + \frac{1}{(\Omega-2\omega_1)^2}\right]$$
$$+ f_{13}f_{32}\left[\frac{1}{(\Omega+\omega_1)^2-\omega_3^2} + \frac{1}{(\Omega-\omega_1)^2-\omega_3^2}\right] \tag{7.132}$$

$$\alpha_2 = \frac{f_{12}(f_{11}+f_{22})}{\Omega(\Omega-2\omega_1)} + \frac{f_{12}^2}{\Omega^2(\Omega-2\omega_1)^2} + \frac{f_{13}f_{32}}{(\Omega-\omega_1)^2-\omega_3^2} \tag{7.133}$$

Substituting for g_1, g_2, and g_3 from (7.128), (7.123), and (7.129) into (7.111)–(7.113), we obtain

$$\dot{\xi}_1 = i\omega_1\xi_1 - \epsilon\mu_1\xi_1 + \frac{i\epsilon}{2\omega_1}\left(\alpha_1 z\bar{z}\xi_2 + \alpha_2 z^2\bar{\xi}_2\right) + \cdots \tag{7.134}$$

$$\dot{\xi}_2 = i\omega_1\xi_2 - \epsilon\mu_2\xi_2 + \frac{i\epsilon}{2\omega_1}\xi_1 + \cdots \tag{7.135}$$

$$\dot{\xi}_3 = i\omega_3\xi_3 - \epsilon\mu_3\xi_3 \tag{7.136}$$

Equations 7.132–7.136 agree with those obtained by Nayfeh and Mook (1979) by using the method of multiple scales.

7.4 Gyroscopic Systems

For simplicity, we consider a two-degree-of-freedom system to illustrate the method of solution. Specifically, we consider

$$\ddot{u}_1 + \lambda_1\dot{u}_2 + \alpha_1 u_1 = \epsilon F_1 = 2\epsilon\,(f_{11}u_1 + f_{12}u_2)\cos\Omega t \tag{7.137}$$

$$\ddot{u}_2 - \lambda_2\dot{u}_1 + \alpha_2 u_2 = \epsilon F_2 = 2\epsilon\,(f_{21}u_1 + f_{22}u_2)\cos\Omega t \tag{7.138}$$

where ϵ, Ω, λ_m, α_m, and f_{mn} are constants.

As a first step, we cast (7.137) and (7.138) in complex-valued form. To accomplish this, we first write down the solution of the unperturbed problem; that is,

$$u_1 = A_1e^{i\omega_1 t} + \bar{A}_1e^{-i\omega_1 t} + A_2e^{i\omega_2 t} + \bar{A}_2e^{-i\omega_2 t} \tag{7.139}$$

$$\dot{u}_1 = i\omega_1\left(A_1e^{i\omega_1 t} - \bar{A}_1e^{-i\omega_1 t}\right) + i\omega_2\left(A_2e^{i\omega_2 t} - \bar{A}_2e^{-i\omega_2 t}\right) \tag{7.140}$$

$$u_2 = \frac{i(\alpha_1-\omega_1^2)}{\lambda_1\omega_1}\left(A_1e^{i\omega_1 t} - \bar{A}_1e^{-i\omega_1 t}\right) + \frac{i(\alpha_1-\omega_2^2)}{\lambda_1\omega_2}\left(A_2e^{i\omega_2 t} - \bar{A}_2e^{-i\omega_2 t}\right) \tag{7.141}$$

$$\dot{u}_2 = -\frac{\alpha_1-\omega_1^2}{\lambda_1}\left(A_1e^{i\omega_1 t} + \bar{A}_1e^{-i\omega_1 t}\right) - \frac{\alpha_1-\omega_2^2}{\lambda_1}\left(A_2e^{i\omega_2 t} + \bar{A}_2e^{-i\omega_2 t}\right) \tag{7.142}$$

where ω_1^2 and ω_2^2 are solutions of

$$\omega^4 - (\alpha_1 + \alpha_2 + \lambda_1\lambda_2)\,\omega^2 + \alpha_1\alpha_2 = 0 \tag{7.143}$$

Here, we assume that ω_1^2 and ω_2^2 are positive and that $\omega_2 > \omega_1$.

We associate $A_1 e^{i\omega_1 t}$ with ζ_1 and $A_2 e^{i\omega_2 t}$ with ζ_2 in (7.139)–(7.142) to define the transformation

$$u_1 = \zeta_1 + \bar{\zeta}_1 + \zeta_2 + \bar{\zeta}_2 \tag{7.144}$$

$$\dot{u}_1 = i\omega_1\left(\zeta_1 - \bar{\zeta}_1\right) + i\omega_2\left(\zeta_2 - \bar{\zeta}_2\right) \tag{7.145}$$

$$u_2 = \frac{i(\alpha_1 - \omega_1^2)}{\lambda_1\omega_1}\left(\zeta_1 - \bar{\zeta}_1\right) + \frac{i(\alpha_1 - \omega_2^2)}{\lambda_1\omega_2}\left(\zeta_2 - \bar{\zeta}_2\right) \tag{7.146}$$

$$\dot{u}_2 = -\frac{\alpha_1 - \omega_1^2}{\lambda_1}\left(\zeta_1 + \bar{\zeta}_1\right) - \frac{\alpha_1 - \omega_2^2}{\lambda_1}\left(\zeta_2 + \bar{\zeta}_2\right) \tag{7.147}$$

It follows from (7.144)–(7.147) that

$$\left(\omega_2^2 - \omega_1^2\right)\left(\zeta_1 + \bar{\zeta}_1\right) = \left(\omega_2^2 - \alpha_1\right)u_1 - \lambda_1\dot{u}_2 \tag{7.148}$$

$$\left(\omega_2^2 - \omega_1^2\right)\left(\zeta_2 + \bar{\zeta}_2\right) = \left(\alpha_1 - \omega_1^2\right)u_1 + \lambda_1\dot{u}_2 \tag{7.149}$$

$$\alpha_1\left(\omega_2^2 - \omega_1^2\right)\left(\zeta_1 - \bar{\zeta}_1\right) = i\omega_1\left[\left(\alpha_1 - \omega_2^2\right)\dot{u}_1 - \lambda_1\omega_2^2 u_2\right] \tag{7.150}$$

$$\alpha_1\left(\omega_2^2 - \omega_1^2\right)\left(\zeta_2 - \bar{\zeta}_2\right) = -i\omega_2\left[\left(\alpha_1 - \omega_1^2\right)\dot{u}_1 - \lambda_1\omega_1^2 u_2\right] \tag{7.151}$$

Solving (7.148) and (7.150) for ζ_1 yields

$$\begin{aligned} 2\alpha_1\left(\omega_2^2 - \omega_1^2\right)\zeta_1 = {} & i\omega_1\left(\alpha_1 - \omega_2^2\right)\dot{u}_1 - \alpha_1\lambda_1\dot{u}_2 \\ & - \alpha_1\left(\alpha_1 - \omega_2^2\right)u_1 - i\lambda_1\omega_1\omega_2^2 u_2 \end{aligned} \tag{7.152}$$

Solving (7.149) and (7.151) for ζ_2 yields

$$\begin{aligned} 2\alpha_1\left(\omega_2^2 - \omega_1^2\right)\zeta_2 = {} & -i\omega_2\left(\alpha_1 - \omega_1^2\right)\dot{u}_1 + \alpha_1\lambda_1\dot{u}_2 \\ & + \alpha_1\left(\alpha_1 - \omega_1^2\right)u_1 + i\lambda_1\omega_1^2\omega_2 u_2 \end{aligned} \tag{7.153}$$

Differentiating (7.152) with respect to t and using (7.137) and (7.138) to eliminate $\ddot{u}_1$ and $\ddot{u}_2$, we obtain

$$\begin{aligned} 2\alpha_1\left(\omega_2^2 - \omega_1^2\right)\dot{\zeta}_1 = {} & -i\omega_1\alpha_1\left[\lambda_1\dot{u}_2 + \left(\alpha_1 - \omega_2^2\right)u_1\right] \\ & - \alpha_1\left[\left(\alpha_1 + \lambda_1\lambda_2 - \omega_2^2\right)\dot{u}_1 - \lambda_1\alpha_2 u_2\right] \\ & + i\epsilon\omega_1\left(\alpha_1 - \omega_2^2\right)F_1 - \epsilon\alpha_1\lambda_1 F_2 \end{aligned} \tag{7.154}$$

It follows from (7.143) that

$$\alpha_1 + \lambda_1\lambda_2 - \omega_2^2 = \omega_1^2 - \alpha_2 \quad \text{and} \quad \alpha_1\alpha_2 = \omega_1^2\omega_2^2 \tag{7.155}$$

Using (7.155), we rewrite (7.154) as

$$2\alpha_1\left(\omega_2^2-\omega_1^2\right)\dot{\zeta}_1 = -i\omega_1\alpha_1\left[\lambda_1\dot{u}_2+\left(\alpha_1-\omega_2^2\right)u_1\right] + \left[\left(\omega_1^2\omega_2^2-\alpha_1\omega_1^2\right)\dot{u}_1+\lambda_1\omega_1^2\omega_2^2u_2\right] + i\epsilon\omega_1\left(\alpha_1-\omega_2^2\right)F_1-\epsilon\alpha_1\lambda_1F_2 \tag{7.156}$$

Using (7.144)–(7.147) to eliminate the u_m and $\dot{u}_m$ from (7.156) and rearranging, we obtain

$$\dot{\zeta}_1 = i\omega_1\zeta_1+\frac{i\epsilon\omega_1(\alpha_1-\omega_2^2)}{2\alpha_1(\omega_2^2-\omega_1^2)}F_1-\frac{\epsilon\lambda_1}{2(\omega_2^2-\omega_1^2)}F_2 \tag{7.157}$$

Differentiating (7.153) and using a procedure similar to that used to determine $\dot{\zeta}_1$, we obtain

$$\dot{\zeta}_2 = i\omega_2\zeta_2-\frac{i\epsilon\omega_2(\alpha_1-\omega_1^2)}{2\alpha_1(\omega_2^2-\omega_1^2)}F_1+\frac{\epsilon\lambda_1}{2(\omega_2^2-\omega_1^2)}F_2 \tag{7.158}$$

Finally, we substitute for F_1 and F_2 from (7.137) and (7.138) into (7.157) and (7.158), use (7.3), (7.144), and (7.146), and obtain

$$\dot{\zeta}_1 = i\omega_1\zeta_1+\frac{i\epsilon\omega_1(\alpha_1-\omega_2^2)}{2\alpha_1(\omega_2^2-\omega_1^2)}(z+\bar{z})\Lambda_1-\frac{\epsilon\lambda_1}{2(\omega_2^2-\omega_1^2)}(z+\bar{z})\Lambda_2 \tag{7.159}$$

$$\dot{\zeta}_2 = i\omega_2\zeta_2-\frac{i\epsilon\omega_2(\alpha_1-\omega_1^2)}{2\alpha_1(\omega_2^2-\omega_1^2)}(z+\bar{z})\Lambda_1+\frac{\epsilon\lambda_1}{2(\omega_2^2-\omega_1^2)}(z+\bar{z})\Lambda_2 \tag{7.160}$$

where

$$\Lambda_1 = f_{11}\left(\zeta_1+\bar{\zeta}_1+\zeta_2+\bar{\zeta}_2\right) + i\,f_{12}\left[\frac{\alpha_1-\omega_1^2}{\lambda_1\omega_1}\left(\zeta_1-\bar{\zeta}_1\right)+\frac{\alpha_1-\omega_2^2}{\lambda_1\omega_2}\left(\zeta_2-\bar{\zeta}_2\right)\right]$$

$$\Lambda_2 = f_{21}\left(\zeta_1+\bar{\zeta}_1+\zeta_2+\bar{\zeta}_2\right) + i\,f_{22}\left[\frac{\alpha_1-\omega_1^2}{\lambda_1\omega_1}\left(\zeta_1-\bar{\zeta}_1\right)+\frac{\alpha_1-\omega_2^2}{\lambda_1\omega_2}\left(\zeta_2-\bar{\zeta}_2\right)\right]$$

To simplify (7.159) and (7.160), we introduce the near-identity transformation (7.44) and choose the h_m to eliminate the nonresonance terms, thereby leaving the resonance and near-resonance terms. To $O(\epsilon)$, there are several resonance combinations of Ω, ω_1, and ω_2. These include:

- $\Omega \approx 2\omega_1$: Principal parametric resonance of the first mode,
- $\Omega \approx 2\omega_2$: Principal parametric resonance of the second mode,
- $\Omega \approx \omega_2+\omega_1$: Combination parametric resonance of the additive type,
- $\Omega \approx \omega_2-\omega_1$: Combination parametric resonance of the difference type.

Next, we discuss the cases $\Omega \approx 2\omega_1$ and $\Omega \approx \omega_2-\omega_1$ treated by Nayfeh and Mook (1979) using the method of multiple scales.

7.4.1
The Case of Ω Near $2\omega_1$

When $\Omega \approx 2\omega_1$ and there are no other resonances, there are no resonance terms in (7.160) and $z\bar{\zeta}_1$ is a near-resonance term in (7.159). Therefore, substituting (7.44) into (7.159) and (7.160) and choosing the h_m to eliminate the nonresonance terms, we obtain the normal form

$$\dot{\eta}_1 = i\omega_1\eta_1 - \frac{\epsilon}{2(\omega_2^2 - \omega_1^2)}\Bigg\{\lambda_1 f_{21} + \lambda_2 f_{12} - i\left[\frac{\omega_1(\alpha_1 - \omega_2^2) f_{11}}{\alpha_1} + \frac{\alpha_1 - \omega_1^2}{\omega_1} f_{22}\right]\Bigg\} \bar{\eta}_1 z \tag{7.161}$$

$$\dot{\eta}_2 = i\omega_2\eta_2 \tag{7.162}$$

Putting $\eta_n = A_n e^{\omega_n t}$ in (7.161) and (7.162) yields the same equations obtained by using the method of multiple scales (Nayfeh and Mook, 1979).

7.4.2
The Case of Ω Near $\omega_2 - \omega_1$

In this case, the resonance terms are $\bar{z}\zeta_2$ and $z\zeta_1$ in (7.159) and (7.160), respectively. Therefore, substituting (7.44) into (7.159) and (7.160) and choosing the h_m to eliminate the nonresonance terms, we obtain the normal forms

$$\dot{\eta}_1 = i\omega_1\eta_1 - \frac{\epsilon}{2(\omega_2^2 - \omega_1^2)}\Bigg\{\lambda_1 f_{21} + \frac{\omega_1(\alpha_1 - \omega_2^2)^2}{\lambda_1\alpha_1\omega_2} f_{12} + i\left(\alpha_1 - \omega_2^2\right)\left(\frac{f_{22}}{\omega_2} - \frac{\omega_1 f_{11}}{\alpha_1}\right)\Bigg\} \eta_2\bar{z} \tag{7.163}$$

$$\dot{\eta}_2 = i\omega_2\eta_2 + \frac{\epsilon}{2(\omega_2^2 - \omega_1^2)}\Bigg\{\lambda_1 f_{21} + \frac{\omega_2(\alpha_1 - \omega_1^2)^2}{\lambda_1\alpha_1\omega_1} f_{12} - i\left(\alpha_1 - \omega_1^2\right)\left(\frac{\omega_2 f_{11}}{\alpha_1} - \frac{f_{22}}{\omega_1}\right)\Bigg\} \eta_1 z \tag{7.164}$$

Putting $\eta_m = A_m e^{i\omega_m t}$ in (7.163) and (7.164) yields the same equations obtained by Nayfeh and Mook (1979) by using the method of multiple scales.

7.5
A Nonlinear Single-Degree-of-Freedom System

Finally, we illustrate the method of normal forms using a parametrically excited single-degree-of-freedom system with quadratic and cubic nonlinearities. Specifically, we conside

$$\ddot{u} + \omega^2 u + 2\epsilon\mu\dot{u} + 2\epsilon F u \cos\Omega t + \epsilon\delta u^2 + \epsilon^2\alpha u^3 = 0 \tag{7.165}$$

Using the transformation (7.2) and (7.3), we rewrite (7.165) as

$$\dot{\zeta} = i\omega\zeta - \epsilon\mu\left(\zeta - \bar{\zeta}\right) + \frac{i\epsilon F}{2\omega}\left(\zeta + \bar{\zeta}\right)(z + \bar{z}) + \frac{i\epsilon\delta}{2\omega}\left(\zeta + \bar{\zeta}\right)^2 + \frac{i\epsilon^2\alpha}{2\omega}\left(\zeta + \bar{\zeta}\right)^3 \tag{7.166}$$

We seek a second-order uniform expansion of the solution of (7.166) in the form (7.6) and (7.7), equate coefficients of like powers of ϵ, and obtain

$$g_1 + \mathcal{L}(h_1) = -\mu\left(\eta - \bar{\eta}\right) + \frac{iF}{2\omega}\left(\eta + \bar{\eta}\right)(z + \bar{z}) + \frac{i\delta}{2\omega}\left(\eta + \bar{\eta}\right)^2 \tag{7.167}$$

$$g_2 + \mathcal{L}(h_2) = -g_1\frac{\partial h_1}{\partial \eta} - \bar{g}_1\frac{\partial h_1}{\partial \bar{\eta}} - \mu\left(h_1 - \bar{h}_1\right) + \frac{iF}{2\omega}\left(h_1 + \bar{h}_1\right)(z + \bar{z}) + \frac{i\delta}{\omega}\left(\eta + \bar{\eta}\right)\left(h_1 + \bar{h}_1\right) + \frac{i\alpha}{2\omega}\left(\eta + \bar{\eta}\right)^3 \tag{7.168}$$

where the operator $\mathcal{L}$ is defined in (7.10). There are two cases, depending on whether $\Omega \approx 2\omega$ (i.e., principal parametric resonance) or Ω is away from 2ω. We first consider the latter case.

7.5.1
The Case of Ω Away from 2ω

In this case, to first order, $-\mu\eta$ is a resonance term and there are no near-resonance terms. Thus, we choose g_1 to eliminate the resonance term in (7.167); that is,

$$g_1 = -\mu\eta \tag{7.169}$$

Then, the form of the remaining terms on the right-hand side of (7.167) suggests choosing h_1 in the form

$$h_1 = \Gamma_1\bar{\eta} + \Gamma_2\eta z + \Gamma_3\bar{\eta}\bar{z} + \Gamma_4\eta\bar{z} + \Gamma_5\bar{\eta}z + \Gamma_6\eta^2 + \Gamma_7\eta\bar{\eta} + \Gamma_8\bar{\eta}^2 \tag{7.170}$$

Substituting (7.170) into (7.167) and using (7.169) and (7.10) yields

$$\begin{aligned}&(2i\omega\Gamma_1 + \mu)\,\bar{\eta} - i\left(\Omega\Gamma_2 - \frac{F}{2\omega}\right)\eta z + i\left((2\omega + \Omega)\,\Gamma_3 + \frac{F}{2\omega}\right)\bar{\eta}\bar{z}\\ &+ i\left(\Omega\Gamma_4 + \frac{F}{2\omega}\right)\eta\bar{z} + i\left((2\omega - \Omega)\,\Gamma_5 + \frac{F}{2\omega}\right)\bar{\eta}z\\ &- i\omega\left(\Gamma_6 - \frac{\delta}{2\omega^2}\right)\eta^2 + i\omega\left(\Gamma_7 + \frac{\delta}{\omega^2}\right)\eta\bar{\eta} + i\omega\left(3\Gamma_8 + \frac{\delta}{2\omega^2}\right)\bar{\eta}^2\\ &= 0\end{aligned} \tag{7.171}$$

Choosing the Γ_m to eliminate the nonresonance terms, we have

$$\Gamma_1 = \frac{i\mu}{2\omega}, \quad \Gamma_2 = \frac{F}{2\omega\Omega}, \quad \Gamma_3 = -\frac{F}{2\omega(\Omega + 2\omega)}, \quad \Gamma_4 = -\frac{F}{2\omega\Omega},$$
$$\Gamma_5 = \frac{F}{2\omega(\Omega - 2\omega)}, \quad \Gamma_6 = \frac{\delta}{2\omega^2}, \quad \Gamma_7 = -\frac{\delta}{\omega^2}, \quad \Gamma_8 = -\frac{\delta}{6\omega^2} \tag{7.172}$$

Substituting (7.172) into (7.170) yields

$$h_1 = \frac{i\mu\bar{\eta}}{2\omega} + \frac{Fz\eta}{2\omega\Omega} - \frac{F\eta\bar{z}}{2\omega\Omega} - \frac{Fz\bar{\eta}}{2\omega(2\omega-\Omega)} - \frac{F\bar{z}\bar{\eta}}{2\omega(2\omega+\Omega)} + \frac{\delta\eta^2}{2\omega^2} - \frac{\delta\eta\bar{\eta}}{\omega^2} - \frac{\delta\bar{\eta}^2}{6\omega^2} \tag{7.173}$$

Substituting (7.173) and (7.6) into (7.2), we have

$$u = \eta + \bar{\eta} - \frac{i\epsilon\mu}{2\omega}(\eta - \bar{\eta}) + \frac{\epsilon\delta}{3\omega^2}(\eta^2 - 3\eta\bar{\eta} + \bar{\eta}^2) + \frac{\epsilon F(z\eta + \bar{\eta}\bar{z})}{\Omega(\Omega+2\omega)} + \frac{\epsilon F(\eta\bar{z} + \bar{\eta}z)}{\Omega(\Omega-2\omega)} + \cdots \tag{7.174}$$

Substituting (7.169) and (7.173) into (7.168) yields

$$\begin{aligned} g_2 + \mathcal{L}(h_2) = -\frac{i\mu^2\eta}{2\omega} + \frac{i}{2\omega}\Bigg\{ & \frac{2F^2\eta z\bar{z}}{\Omega^2 - 4\omega^2} + \frac{F^2\bar{\eta}z^2}{\Omega(\Omega-2\omega)} \\ & + \left[\frac{4}{\Omega^2-4\omega^2} - \frac{2}{\omega^2}\right] F\delta\eta\bar{\eta}z + \left(3\alpha - \frac{10\delta^2}{3\omega^2}\right)\eta^2\bar{\eta} \\ & + \left[\frac{1}{3\omega^2} + \frac{2}{\Omega(\Omega-2\omega)}\right] F\delta\left(\eta^2\bar{z} + \bar{\eta}^2 z\right)\Bigg\} + \text{NRT}\,. \end{aligned} \tag{7.175}$$

There are two possible resonances at this order: those corresponding to $\Omega \approx \omega$ (fundamental parametric resonance) and $\Omega \approx 3\omega$ (subharmonic resonance of order one-third).

When Ω is away from ω and 3ω, choosing g_2 to eliminate the resonance terms in (7.175), we obtain

$$g_2 = -\frac{i\mu^2\eta}{2\omega} + \frac{i}{2\omega}\left\{\frac{2F^2\eta z\bar{z}}{\Omega^2 - 4\omega^2} + \left(3\alpha - \frac{10\delta^2}{3\omega^2}\right)\eta^2\bar{\eta}\right\} \tag{7.176}$$

Substituting (7.169) and (7.176) into (7.7), we obtain the normal form

$$\dot{\eta} = i\omega\eta - \epsilon\mu\eta - \frac{i\epsilon^2\mu^2}{2\omega}\eta + \frac{i\epsilon^2F^2\eta z\bar{z}}{\omega(\Omega^2 - 4\omega^2)} + i\epsilon^2\left(\frac{3\alpha}{2\omega} - \frac{5\delta^2}{3\omega^3}\right)\eta^2\bar{\eta} \tag{7.177}$$

When $\Omega \approx \omega$, choosing g_2 to eliminate the resonance and near-resonance terms from (7.175) and approximating Ω with ω, we obtain

$$g_2 = -\frac{i\mu^2\eta}{2\omega} - \frac{i}{2\omega}\left\{\frac{2F^2}{3\omega^2}\eta z\bar{z} + \frac{F^2}{\omega^2}\bar{\eta}z^2 + \frac{10F\delta}{3\omega^2}\eta\bar{\eta}z + \frac{5F\delta}{3\omega^2}\eta^2\bar{z} - \left(3\alpha - \frac{10\delta^2}{3\omega^2}\right)\eta^2\bar{\eta}\right\} \tag{7.178}$$

Substituting (7.169) and (7.178) into (7.7), we obtain the normal form

$$\dot{\eta} = i\omega\eta - \epsilon\mu\eta - \frac{i\epsilon^2\mu^2}{2\omega}\eta - \frac{i\epsilon^2}{2\omega}\left\{\frac{2F^2}{3\omega^2}\eta z\bar{z} + \frac{F^2}{\omega^2}\bar{\eta}z^2 + \frac{10F\delta}{3\omega^2}\eta\bar{\eta}z + \frac{5F\delta}{3\omega^2}\eta^2\bar{z} - \left(3\alpha - \frac{10\delta^2}{3\omega^2}\right)\eta^2\bar{\eta}\right\} \tag{7.179}$$

Putting

$$\eta = Ae^{i\omega t} \quad \text{and} \quad z = e^{i\Omega t} \tag{7.180}$$

in (7.179), we obtain the equation found by using the method of multiple scales (Exercise 7.6.2).

When $\Omega \approx 3\omega$, choosing g_2 to eliminate resonance and near-resonance terms in (7.175) and approximating Ω with 3ω, we obtain

$$g_2 = -\frac{i\mu^2\eta}{2\omega} + \frac{i}{2\omega}\left\{\frac{2F^2}{5\omega^2}\eta z\bar{z} + \left(3\alpha - \frac{10\delta^2}{3\omega^2}\right)\eta^2\bar{\eta} + \frac{F\delta}{\omega^2}\bar{\eta}^2 z\right\} \tag{7.181}$$

Substituting (7.169) and (7.181) into (7.7), we obtain the normal form

$$\dot{\eta} = i\omega\eta - \frac{\epsilon^2\mu^2\eta}{2\omega} + \frac{i\epsilon^2}{2\omega}\left\{\frac{2F^2}{5\omega^2}\eta z\bar{z} + \left(3\alpha - \frac{10\delta^2}{3\omega^2}\right)\eta^2\bar{\eta} + \frac{F\delta}{\omega^2}\eta^2 z\right\} \tag{7.182}$$

Substituting (7.180) into (7.182), we obtain the equation found by using the method of multiple scales (Exercise 7.6.2).

7.5.2 The Case of Ω Near 2ω

In this case, it follows from (7.172) that Γ_5 has a small-divisor term and hence $z\bar{\eta}$ is near-resonance in (7.167). Choosing g_1 to eliminate the resonance and near-resonance terms in (7.167), we obtain

$$g_1 = -\mu\eta + \frac{iF}{2\omega}\bar{\eta}z \tag{7.183}$$

Then, we can choose all of the Γ_m, $m = 1, 2, \ldots, 8$, except Γ_5, as in (7.172) to eliminate the nonresonance terms. Consequently,

$$h_1 = \frac{i\mu\bar{\eta}}{2\omega} + \frac{Fz\eta}{2\omega\Omega} - \frac{F\eta\bar{z}}{2\omega\Omega} - \frac{F\bar{z}\bar{\eta}}{2\omega(2\omega+\Omega)} + \frac{\delta\eta^2}{2\omega^2} - \frac{\delta\eta\bar{\eta}}{\omega^2} - \frac{\delta\bar{\eta}^2}{6\omega^2} \tag{7.184}$$

Substituting (7.184) into (7.2) yields

$$u = \eta + \bar{\eta} - \frac{i\epsilon\mu}{2\omega}(\eta - \bar{\eta}) + \frac{\epsilon\delta}{3\omega^2}(\eta^2 - 3\eta\bar{\eta} + \bar{\eta}^2) + \frac{\epsilon F(z\eta + \bar{\eta}\bar{z})}{\Omega(\Omega + 2\omega)} - \frac{\epsilon F(\eta\bar{z} + \bar{\eta}z)}{2\omega\Omega} + \cdots \tag{7.185}$$

Substituting (7.183) and (7.184) into (7.168) and choosing g_2 to eliminate the resonance and near-resonance terms, we obtain

$$g_2 = -\frac{i\mu^2}{2\omega}\eta - \frac{(\Omega + 2\omega)\mu F}{4\omega^2\Omega}z\bar{\eta} - \frac{iF^2}{4\omega^2(\Omega + 2\omega)}\eta z\bar{z} + \frac{i}{2\omega}\left(3\alpha - \frac{10\delta^2}{3\omega^2}\right)\eta^2\bar{\eta} \tag{7.186}$$

Substituting (7.183) and (7.186) into (7.7), we obtain the normal form

$$\dot{\eta} = i\omega\eta - \epsilon\mu\eta + \frac{i\epsilon F}{2\omega}\bar{\eta}z - \frac{i\epsilon^2\mu^2}{2\omega}\eta - \frac{(\Omega + 2\omega)\epsilon^2\mu F}{4\omega^2\Omega}z\bar{\eta} - \frac{i\epsilon^2 F^2}{4\omega^2(\Omega + 2\omega)}\eta z\bar{z} + \frac{i\epsilon^2}{2\omega}\left(3\alpha - \frac{10\delta^2}{3\omega^2}\right)\eta^2\bar{\eta} \tag{7.187}$$

7.6 Exercises

7.6.1 Use the method of multiple scales to determine the normal form of (7.1) when $\Omega \approx \omega$ and $\Omega \approx 2\omega$.

Use the complex-valued form (7.5) and seek a second-order approximate solution in the form

$$\zeta = \zeta_1(T_0, T_1, T_2) + \epsilon\zeta_2(T_0, T_1, T_2) + +\epsilon^3\zeta_2(T_0, T_1, T_2) + \cdots$$

and obtain the system of equations

$$D_0\zeta_1 - i\omega\zeta_1 = 0$$

$$D_0\zeta_2 - i\omega\zeta_2 = -D_1\zeta_1 - \mu(\zeta_1 - \bar{\zeta}_1) + \frac{i}{2\omega}\left(e^{iT_0\Omega} + e^{-iT_0\Omega}\right)(\zeta_1 + \bar{\zeta}_1)$$

$$D_0\zeta_3 - i\omega\zeta_3 = -D_1\zeta_2 - D_2\zeta_1 - \mu(\zeta_2 - \bar{\zeta}_2) + \frac{i}{2\omega}\left(e^{iT_0\Omega} + e^{-iT_0\Omega}\right)(\zeta_2 + \bar{\zeta}_2)$$

Express the solution of the first-order problem as

$$\zeta_1 = A(T_1, T_2)e^{i\omega T_0}$$

Then, show that the second-order problem becomes

$$D_0\zeta_2 - i\omega\zeta_2 = -D_1Ae^{i\omega T_0} - \mu Ae^{i\omega T_0} + \mu\bar{A}e^{-i\omega T_0} + \frac{i}{2\omega}\left(e^{iT_0\Omega} + e^{-iT_0\Omega}\right)\left(Ae^{i\omega T_0} + \bar{A}e^{-i\omega T_0}\right)$$

The Case $\Omega \approx \omega$ Let $\Omega = \omega + \epsilon^2\sigma$ and show that the solvability condition of the second-order problem is

$$D_1 A = -\mu A$$

Then, show that

$$\zeta_2 = \frac{i\mu}{2\omega}\bar{A}e^{-i\omega T_0} + \frac{1}{2\omega\Omega}Ae^{i(\Omega+\omega)T_0} - \frac{1}{2\omega\Omega}Ae^{i(\omega-\Omega)T_0} - \frac{1}{2\omega(\Omega+2\omega)}\bar{A}e^{-i(\Omega+\omega)T_0} + \frac{1}{2\omega(\Omega-2\omega)}\bar{A}e^{i(\Omega-\omega)T_0}$$

Substitute for ζ_1 and ζ_2 into the third-order equation, eliminate the terms that produce secular terms, and obtain

$$D_2 A = -\frac{i\mu^2}{2\omega}A + \frac{i}{\omega(\Omega^2-4\omega^2)}A + \frac{i}{2\omega\Omega(\Omega-2\omega)}\bar{A}e^{2i\sigma T_2}$$

Finally, use the method of reconstitution to obtain the normal form

$$\dot{A} = -\epsilon\mu A - \epsilon^2\left[\frac{i\mu^2}{2\omega}A - \frac{i}{\omega(\Omega^2-4\omega^2)}A - \frac{i}{2\omega\Omega(\Omega-2\omega)}\bar{A}e^{2i\sigma T_2}\right]$$

Compare these results with those obtained in Section 7.1.1 by using the method of normal forms.

The Case $\Omega \approx 2\omega$ Let $\Omega = 2\omega + \epsilon\sigma$ and show that the solvability condition of the second-order problem is

$$D_1 A = -\mu A + \frac{i}{2\omega}\bar{A}e^{i\sigma T_1}$$

Then, show that

$$\zeta_2 = \frac{i\mu}{2\omega}\bar{A}e^{-i\omega T_0} + \frac{1}{2\omega\Omega}e^{i(\Omega+\omega)T_0} - \frac{1}{2\omega\Omega}e^{i(\omega-\Omega)T_0} - \frac{1}{2\omega(\Omega+2\omega)}e^{-i(\Omega+\omega)T_0}$$

Substitute for ζ_1 and ζ_2 into the third-order equation, eliminate the terms that produce secular terms, and obtain

$$D_2 A = -\frac{i\mu^2}{2\omega}A - \frac{i}{4\omega^2\Omega(\Omega+2\omega)}A - \frac{(\Omega+2\omega)\mu}{4\omega^2\Omega}\bar{A}e^{i\sigma T_1}$$

Finally, use the method of reconstitution to obtain the normal form

$$\dot{A} = -\epsilon\left[\mu A - \frac{i}{2\omega}\bar{A}e^{i\sigma T_1}\right] - \epsilon^2\left[\frac{i\mu^2}{2\omega}A + \frac{i}{4\omega^2\Omega(\Omega+2\omega)}A + \frac{(\Omega+2\omega)\mu}{4\omega^2\Omega}\bar{A}e^{i\sigma T_1}\right]$$

Compare these results with those obtained in Section 7.1.2 by using the method of normal forms.

7.6.2 Use the method of multiple scales to determine the normal form of (7.165) when (a) Ω is away from $\omega, 2\omega$, and 3ω; (b) $\Omega \approx \omega$; (c) $\Omega \approx 3\omega$; and (d) $\Omega \approx 2\omega$.
Seek an approximate solution of (7.166) in the form

$$\zeta = \zeta_1(T_0, T_1, T_2) + \epsilon\zeta_2(T_0, T_1, T_2) + +\epsilon^3\zeta_2(T_0, T_1, T_2) + \cdots$$

and obtain the system of equations

$$D_0\zeta_1 - i\omega\zeta_1 = 0$$

$$D_0\zeta_2 - i\omega\zeta_2 = -D_1\zeta_1 - \mu(\zeta_1 - \bar{\zeta}_1) + \frac{i\delta}{2\omega}(\zeta_1 + \bar{\zeta}_1)^2 + \frac{iF}{2\omega}\left(e^{i\Omega T_0} + e^{-i\Omega T_0}\right)(\zeta_1 + \bar{\zeta}_1)$$

$$D_0\zeta_3 - i\omega\zeta_3 = -D_1\zeta_2 - D_2\zeta_1 - \mu(\zeta_2 - \bar{\zeta}_2) + \frac{i\delta}{\omega}(\zeta_1 + \bar{\zeta}_1)(\zeta_2 + \bar{\zeta}_2) + \frac{i\alpha}{2\omega}(\zeta_1 + \bar{\zeta}_1)^3 + \frac{iF}{2\omega}\left(e^{i\Omega T_0} + e^{-i\Omega T_0}\right)(\zeta_2 + \bar{\zeta}_2)$$

Express the solution of the first-order problem as

$$\zeta_1 = A(T_1, T_2)e^{i\omega T_0}$$

Then, show that the second-order problem becomes

$$D_0\zeta_2 - i\omega\zeta_2 = -D_1Ae^{i\omega T_0} - \mu Ae^{i\omega T_0} + \frac{i\delta}{2\omega}\left(Ae^{i\omega T_0} + \bar{A}e^{-i\omega T_0}\right) + \mu\bar{A}e^{-i\omega T_0} + \frac{iF}{2\omega}\left(e^{i\Omega T_0} + e^{-i\Omega T_0}\right)\left(Ae^{i\omega T_0} + \bar{A}e^{-i\omega T_0}\right)$$

The Case Ω Away from 2ω When Ω is away from 2ω, there are no near-resonances at second order. Eliminate the secular term and obtain

$$D_1A = -\mu A$$

Then, show that

$$\zeta_2 = \frac{i\mu}{2\omega}\bar{A}e^{-i\omega T_0} + \frac{\delta}{2\omega^2}A^2e^{2i\omega T_0} - \frac{\delta}{\omega^2}A\bar{A} - \frac{\delta}{6\omega^2}\bar{A}^2e^{-2i\omega T_0} + \frac{F}{2\omega\Omega}Ae^{i(\Omega+\omega)T_0} - \frac{F}{2\omega\Omega}Ae^{i(\omega-\Omega)T_0} - \frac{F}{2\omega(\Omega+2\omega)}\bar{A}e^{-i(\Omega+\omega)T_0} + \frac{F}{2\omega(\Omega-2\omega)}\bar{A}e^{i(\Omega-\omega)T_0}$$

Substitute ζ_1 and ζ_2 into the third-order equation, use the expression for D_1A, eliminate the terms that lead to secular and small-divisor terms, and obtain

$$D_2A = -\frac{i\mu^2}{2\omega}A + \frac{iF^2}{\omega(\Omega^2 - 4\omega^2)}A + i\left(\frac{3\alpha}{2\omega} - \frac{5\delta^2}{3\omega^3}\right)A^2\bar{A}$$

when Ω is away from ω and 3ω,

$$D_2A = -\frac{i\mu^2}{2\omega}A + \frac{iF^2}{\omega(\Omega^2 - 4\omega^2)}A + i\left(\frac{3\alpha}{2\omega} - \frac{5\delta^2}{3\omega^3}\right)A^2\bar{A} + \frac{5iF\delta}{6\omega^3}A^2e^{-i\sigma T_2} - \frac{iF\delta}{3\omega^3}A\bar{A}e^{i\sigma T_2}$$

when $\Omega \approx \omega$, and

$$D_2A = -\frac{i\mu^2}{2\omega}A + \frac{iF^2}{5\omega^3}A + i\left(\frac{3\alpha}{2\omega} - \frac{5\delta^2}{3\omega^3}\right)A^2\bar{A} + \frac{iF\delta}{2\omega^3}\bar{A}^2e^{i\sigma T_2}$$

when $\Omega \approx 3\omega$.

Compare the results obtained with the method of multiple scales with those obtained in Section 7.5.1 with the method of normal forms.

The Case $\Omega \approx 2\omega$ Eliminate the terms that produce secular and small-divisor terms from the second-order equation and obtain

$$D_1A = -\mu A + \frac{iF}{2\omega}\bar{A}e^{i\sigma T_1}$$

Then, show that

$$\zeta_2 = \frac{i\mu}{2\omega}\bar{A}e^{-i\omega T_0} + \frac{\delta}{2\omega^2}A^2e^{2i\omega T_0} - \frac{\delta}{\omega^2}A\bar{A} - \frac{\delta}{6\omega^2}\bar{A}^2e^{-2i\omega T_0} + \frac{F}{2\omega\Omega}Ae^{i(\Omega+\omega)T_0} - \frac{F}{2\omega\Omega}Ae^{i(\omega-\Omega)T_0} - \frac{F}{2\omega(\Omega+2\omega)}\bar{A}e^{-i(\Omega+\omega)T_0}$$

Substitute ζ_1 and ζ_2 into the third-order equation, use the expression for D_1A, eliminate the terms that lead to secular and small-divisor terms, and obtain

$$D_2A = -\frac{i\mu^2}{2\omega}A - \frac{iF^2}{4\omega^2(\Omega+2\omega)}A + i\left(\frac{3\alpha}{2\omega} - \frac{5\delta^2}{3\omega^3}\right)A^2\bar{A} - \frac{F\mu(\Omega+2\omega)}{4\omega^2\Omega}\bar{A}e^{i\sigma T_1}$$

Compare the results obtained with the method of multiple scales with those obtained in Section 7.5.2 with the method of normal forms.

7.6.3 Consider the equation

$$\ddot{u} + \omega_0^2u + 2\epsilon u^3\cos 2t = 0 \quad \epsilon \ll 1$$

Use the methods of multiple scales and normal forms to determine the equations describing the amplitude and the phase to first order when

a) ω_0 is away from 1 and 1/2,
b) $\omega_0 \approx 1$,
c) $\omega_0 \approx 1/2$.

7.6.4 The parametric excitation of a two-degree-of-freedom system is governed

$$\ddot{u}_1 + \omega_1^2 u_1 + \epsilon \cos \Omega t \, (f_{11} u_1 + f_{12} u_2) = 0$$
$$\ddot{u}_2 + \omega_2^2 u_2 + \epsilon \cos \Omega t \, (f_{21} u_2 + f_{22} u_2) = 0$$

Use the methods of multiple scales and normal forms to determine the equations describing the amplitudes and the phases when $\Omega \approx \omega_2 \mp \omega_1$.

8
MDOF Systems with Quadratic Nonlinearities

In this chapter, we treat multiple-degree-of-freedom nongyroscopic and gyroscopic systems with quadratic nonlinearities subject to harmonic excitations.

8.1
Nongyroscopic Systems

A general nonlinear multiple-degree-of-freedom nongyroscopic system can be modeled by

$$\ddot{\boldsymbol{x}} + A\boldsymbol{x} + D\dot{\boldsymbol{x}} + \boldsymbol{N}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \boldsymbol{t}) = 0 \tag{8.1}$$

where A and D are $n \times n$ matrices, $\boldsymbol{x}$ is a column vector of length n, and $\boldsymbol{N}$ is a nonlinear vector function of $\boldsymbol{x}$, $\dot{\boldsymbol{x}}$, and t having n components.

In this section, we treat systems whose linear parts are expressed in normal-mode form, and in Section 8.3 we treat two linearly coupled oscillators. The linear part of (8.1) can be put in the following normal-mode form by using the linear transformation $\boldsymbol{x} = P\boldsymbol{u}$:

$$\ddot{\boldsymbol{u}} + J\boldsymbol{u} + \hat{D}\dot{\boldsymbol{u}} + \hat{\boldsymbol{N}}(\boldsymbol{u}, \dot{\boldsymbol{u}}, \boldsymbol{t}) = 0 \tag{8.2}$$

where

$$J = P^{-1}AP\,, \quad \hat{D} = P^{-1}DP\,, \quad \hat{\boldsymbol{N}} = P^{-1}\boldsymbol{N}(P\boldsymbol{u}, P\dot{\boldsymbol{u}}, t)$$

We choose P so that J has a Jordan canonical form and assume that its diagonal elements are positive. We also assume modal damping so that $\hat{D}$ is a diagonal matrix. We limit our discussion to systems for which the frequencies (i.e., eigenvalues of A) are distinct and hence J is diagonal.

We can explore the principal features of the analysis and limit the algebra to a minimum by considering a system having three degrees of freedom. Thus, we consider

$$\ddot{u}_m + \omega_m^2 u_m + 2\epsilon\mu_m\dot{u}_m = \epsilon\frac{\partial V}{\partial u_m}(u_1, u_2, u_3) + 2f_m\cos(\Omega t + \tau_m) \tag{8.3}$$

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where

$$V = \alpha_1 u_1^3 + \alpha_2 u_1^2 u_2 + \alpha_3 u_1 u_2^2 + \alpha_4 u_1^2 u_3 + \alpha_5 u_1 u_3^2 + \alpha_6 u_2^3 + \alpha_7 u_2^2 u_3 + \alpha_8 u_2 u_3^2 + \alpha_9 u_3^3 + \alpha_{10} u_1 u_2 u_3 \tag{8.4}$$

and the ω_n, α_n, f_n, τ_n, μ_n, and Ω are constants independent of ϵ. Here, ϵ is a small nondimensional parameter used for bookkeeping purposes.

Again, as a first step, we cast (8.3) and (8.4) in complex-valued form using the transformation

$$u_m = \zeta_m + \bar{\zeta}_m\,, \quad \dot{u}_m = i\omega_m\left(\zeta_m - \bar{\zeta}_m\right) \tag{8.5}$$

$$\dot{z}_m = i\Omega_m z_m\,, \quad z_m = f_m e^{i(\Omega t + \tau_m)} \tag{8.6}$$

The result is

$$\dot{\zeta}_m = i\omega_m \zeta_m - \epsilon\mu_m\left(\zeta_m - \bar{\zeta}_m\right) - \frac{i\epsilon}{2\omega_m}\frac{\partial V}{\partial u_m}\left(\zeta_1 + \bar{\zeta}_1, \zeta_2 + \bar{\zeta}_2, \zeta_3 + \bar{\zeta}_3\right) - \frac{i}{2\omega_m}\left(z_m + \bar{z}_m\right) \tag{8.7}$$

To simplify the $O(1)$ terms in (8.7), we introduce the transformation

$$\zeta_m = \eta_m + \Delta_{m1} z_m + \Delta_{m2} \bar{z}_m \tag{8.8}$$

use (8.6), and obtain

$$\dot{\eta}_m = i\omega_m \eta_m + i\left[\left(\omega_m - \Omega\right)\Delta_{m1} - \frac{1}{2\omega_m}\right] z_m + i\left[\left(\omega_m + \Omega\right)\Delta_{m2} - \frac{1}{2\omega_m}\right]\bar{z}_m + O(\epsilon) \tag{8.9}$$

There are two possibilities: (a) $\Omega \approx \omega_m$ for $m = 1$ or 2 or 3; and (b) Ω is away from all of the ω_m. The first possibility corresponds to primary resonance of the mth mode. It is treated in Section 8.1.4.

When Ω is away from all of the ω_m, we choose the Δ_{mn} to eliminate z_m and $\bar{z}_m$ from (8.9); that is,

$$\Delta_{m1} = \frac{1}{2\omega_m(\omega_m - \Omega)} \quad \text{and} \quad \Delta_{m2} = \frac{1}{2\omega_m(\omega_m + \Omega)} \tag{8.10}$$

We note that there are no small divisors in (8.10) because we assumed that Ω is away from every ω_m. Substituting (8.8) into (8.7) and using (8.10), we obtain

$$\begin{aligned}\dot{\eta}_1 = {} & i\omega_1\eta_1 - \epsilon\mu_1(\eta_1 - \bar{\eta}_1) - \frac{i\epsilon}{2\omega_1}\left[3\alpha_1\left(\eta_1 + \bar{\eta}_1 + \frac{z_1 + \bar{z}_1}{\omega_1^2 - \Omega^2}\right)^2\right. \\ & + 2\alpha_2\left(\eta_1 + \bar{\eta}_1 + \frac{z_1 + \bar{z}_1}{\omega_1^2 - \Omega^2}\right)\left(\eta_2 + \bar{\eta}_2 + \frac{z_2 + \bar{z}_2}{\omega_2^2 - \Omega^2}\right) \\ & + 2\alpha_4\left(\eta_1 + \bar{\eta}_1 + \frac{z_1 + \bar{z}_1}{\omega_1^2 - \Omega^2}\right)\left(\eta_3 + \bar{\eta}_3 + \frac{z_3 + \bar{z}_3}{\omega_3^2 - \Omega^2}\right) \\ & + \alpha_3\left(\eta_2 + \bar{\eta}_2 + \frac{z_2 + \bar{z}_2}{\omega_2^2 - \Omega^2}\right)^2 + \alpha_5\left(\eta_3 + \bar{\eta}_3 + \frac{z_3 + \bar{z}_3}{\omega_3^2 - \Omega^2}\right)^2 \\ & \left. + \alpha_{10}\left(\eta_2 + \bar{\eta}_2 + \frac{z_2 + \bar{z}_2}{\omega_2^2 - \Omega^2}\right)\left(\eta_3 + \bar{\eta}_3 + \frac{z_3 + \bar{z}_3}{\omega_3^2 - \Omega^2}\right)\right] \end{aligned} \tag{8.11}$$

$$\begin{aligned}\dot{\eta}_2 = {} & i\omega_2\eta_2 - \epsilon\mu_2(\eta_2 - \bar{\eta}_2) - \frac{i\epsilon}{2\omega_2}\left[\alpha_2\left(\eta_1 + \bar{\eta}_1 + \frac{z_1 + \bar{z}_1}{\omega_1^2 - \Omega^2}\right)^2\right. \\ & + 2\alpha_3\left(\eta_1 + \bar{\eta}_1 + \frac{z_1 + \bar{z}_1}{\omega_1^2 - \Omega^2}\right)\left(\eta_2 + \bar{\eta}_2 + \frac{z_2 + \bar{z}_2}{\omega_2^2 - \Omega^2}\right) \\ & + 2\alpha_7\left(\eta_2 + \bar{\eta}_2 + \frac{z_2 + \bar{z}_2}{\omega_2^2 - \Omega^2}\right)\left(\eta_3 + \bar{\eta}_3 + \frac{z_3 + \bar{z}_3}{\omega_3^2 - \Omega^2}\right) \\ & + 3\alpha_6\left(\eta_2 + \bar{\eta}_2 + \frac{z_2 + \bar{z}_2}{\omega_2^2 - \Omega^2}\right)^2 + \alpha_8\left(\eta_3 + \bar{\eta}_3 + \frac{z_3 + \bar{z}_3}{\omega_3^2 - \Omega^2}\right)^2 \\ & \left. + \alpha_{10}\left(\eta_1 + \bar{\eta}_1 + \frac{z_1 + \bar{z}_1}{\omega_1^2 - \Omega^2}\right)\left(\eta_3 + \bar{\eta}_3 + \frac{z_3 + \bar{z}_3}{\omega_3^2 - \Omega^2}\right)\right] \end{aligned} \tag{8.12}$$

$$\begin{aligned}\dot{\eta}_3 = {} & i\omega_3\eta_3 - \epsilon\mu_3(\eta_3 - \bar{\eta}_3) - \frac{i\epsilon}{2\omega_3}\left[\alpha_4\left(\eta_1 + \bar{\eta}_1 + \frac{z_1 + \bar{z}_1}{\omega_1^2 - \Omega^2}\right)^2\right. \\ & + 2\alpha_5\left(\eta_1 + \bar{\eta}_1 + \frac{z_1 + \bar{z}_1}{\omega_1^2 - \Omega^2}\right)\left(\eta_3 + \bar{\eta}_3 + \frac{z_3 + \bar{z}_3}{\omega_3^2 - \Omega^2}\right) \\ & + 2\alpha_8\left(\eta_2 + \bar{\eta}_2 + \frac{z_2 + \bar{z}_2}{\omega_2^2 - \Omega^2}\right)\left(\eta_3 + \bar{\eta}_3 + \frac{z_3 + \bar{z}_3}{\omega_3^2 - \Omega^2}\right) \\ & + \alpha_7\left(\eta_2 + \bar{\eta}_2 + \frac{z_2 + \bar{z}_2}{\omega_2^2 - \Omega^2}\right)^2 + 3\alpha_9\left(\eta_3 + \bar{\eta}_3 + \frac{z_3 + \bar{z}_3}{\omega_3^2 - \Omega^2}\right)^2 \\ & \left. + \alpha_{10}\left(\eta_1 + \bar{\eta}_1 + \frac{z_1 + \bar{z}_1}{\omega_1^2 - \Omega^2}\right)\left(\eta_2 + \bar{\eta}_2 + \frac{z_2 + \bar{z}_2}{\omega_2^2 - \Omega^2}\right)\right] \end{aligned} \tag{8.13}$$

Because $\dot{z}_n = i\Omega z_n$, resonance occurs when

- $\omega_s \approx 2\omega_m$: Two-to-one autoparametric resonance,
- $\omega_s \approx \omega_n + \omega_m$: Combination autoparametric resonance,

- $\Omega \approx 2\omega_m$: Subharmonic resonance of order one-half,
- $\Omega \approx 1/2\omega_m$: Superharmonic resonance of order two,
- $\Omega \approx \omega_n \pm \omega_m$: Combination resonance.

The first two resonances are called *internal* or *autoparametric resonances*, whereas the last three are called external resonances because they are produced by the excitation. We note that some of these resonances might occur simultaneously. Next, we treat the three autoparametric resonances $\omega_2 \approx 2\omega_1$, $\omega_3 \approx \omega_2 + \omega_1$, and $\omega_2 \approx 2\omega_1$ and $\omega_3 \approx 2\omega_2$ in conjunction with a few of the external resonances.

8.1.1
Two-to-One Autoparametric Resonance

We consider the case $\omega_2 \approx 2\omega_1$ and no other autoparametric resonances. To simplify (8.11)–(8.13), we introduce the near-identity transformation

$$\eta_m = \xi_m + \epsilon h_m\left(\xi_n, \bar{\xi}_n, z_n, \bar{z}_n\right) \tag{8.14}$$

and choose the h_m to eliminate the nonresonance terms. The autoparametric resonance produces the near-resonance terms $\xi_2\bar{\xi}_1$ and ξ_1^2 in (8.11) and (8.12), respectively. The near-resonance terms produced by the excitation depend on the resonances being considered. Next, normal forms corresponding to several of these resonances are presented.

No External Resonance

$$\dot{\xi}_1 = i\omega_1\xi_1 - \epsilon\mu_1\xi_1 - \frac{i\epsilon\alpha_2}{\omega_1}\xi_2\bar{\xi}_1 \tag{8.15}$$

$$\dot{\xi}_2 = i\omega_2\xi_2 - \epsilon\mu_2\xi_2 - \frac{i\epsilon\alpha_2}{2\omega_2}\xi_1^2 \tag{8.16}$$

$$\dot{\xi}_3 = i\omega_3\xi_3 - \epsilon\mu_3\xi_3 \tag{8.17}$$

When expressed in the polar coordinates $\xi_n = 1/2r_n e^{i\beta_n}$, (8.15)–(8.17) become

$$\dot{r}_1 = -\epsilon\mu_1 r_1 + \frac{\epsilon\alpha_2}{2\omega_1} r_1 r_2 \sin\left(\beta_2 - 2\beta_1\right) \tag{8.18}$$

$$\dot{r}_2 = -\epsilon\mu_2 r_2 - \frac{\epsilon\alpha_2}{4\omega_2} r_1^2 \sin\left(\beta_2 - 2\beta_1\right) \tag{8.19}$$

$$\dot{r}_3 = -\epsilon\mu_3 r_3 \tag{8.20}$$

$$\dot{r}_1\dot{\beta}_1 = \omega_1 r_1 - \frac{\epsilon\alpha_2}{2\omega_1} r_1 r_2 \cos\left(\beta_2 - 2\beta_1\right) \tag{8.21}$$

$$\dot{r}_2\dot{\beta}_2 = \omega_2 r_2 - \frac{\epsilon\alpha_2}{4\omega_1} r_1^2 \cos\left(\beta_2 - 2\beta_2\right) \tag{8.22}$$

$$r_3\dot{\beta}_3 = \omega_3 r_3 \tag{8.23}$$

$\Omega \approx 1/2\omega_1$

$$\dot{\xi}_1 = i\omega_1\xi_1 - \epsilon\mu_1\xi_1 - \frac{i\epsilon\alpha_2}{\omega_1}\xi_2\bar{\xi}_1 - \frac{i\epsilon}{2\omega_1}\left[\frac{3\alpha_1 z_1^2}{(\omega_1^2-\Omega^2)^2} + \frac{\alpha_3 z_2^2}{(\omega_2^2-\Omega^2)^2} + \frac{\alpha_5 z_3^2}{(\omega_3^2-\Omega^2)^2} + \frac{2\alpha_2 z_1 z_2}{(\omega_1^2-\Omega^2)(\omega_2^2-\Omega^2)} + \frac{2\alpha_4 z_1 z_3}{(\omega_1^2-\Omega^2)(\omega_3^2-\Omega^2)} + \frac{\alpha_{10} z_2 z_3}{(\omega_2^2-\Omega^2)(\omega_3^2-\Omega^2)}\right] \tag{8.24}$$

$$\dot{\xi}_2 = i\omega_2\xi_2 - \epsilon\mu_2\xi_2 - \frac{i\epsilon\alpha_2}{2\omega_2}\xi_1^2 \tag{8.25}$$

$$\dot{\xi}_3 = i\omega_3\xi_3 - \epsilon\mu_3\xi_3 \tag{8.26}$$

$\Omega \approx 2\omega_2$

$$\dot{\xi}_1 = i\omega_1\xi_1 - \epsilon\mu_1\xi_1 - \frac{i\epsilon\alpha_2}{\omega_1}\xi_2\bar{\xi}_1 \tag{8.27}$$

$$\dot{\xi}_2 = i\omega_2\xi_2 - \epsilon\mu_2\xi_2 - \frac{i\epsilon\alpha_2}{2\omega_2}\xi_1^2 - \frac{i\epsilon}{\omega_2}\left[\frac{\alpha_3 z_1}{\omega_1^2-\Omega^2} + \frac{3\alpha_6 z_2}{\omega_2^2-\Omega^2} + \frac{\alpha_7 z_3}{\omega_3^2-\Omega^2}\right]\bar{\xi}_2 \tag{8.28}$$

$$\dot{\xi}_3 = i\omega_3\xi_3 - \epsilon\mu_3\xi_3 \tag{8.29}$$

$\Omega \approx \omega_3 + \omega_2$

$$\dot{\xi}_1 = i\omega_1\xi_1 - \epsilon\mu_1\xi_1 - \frac{i\epsilon\alpha_2}{\omega_1}\xi_2\bar{\xi}_1 \tag{8.30}$$

$$\dot{\xi}_2 = i\omega_2\xi_2 - \epsilon\mu_2\xi_2 - \frac{i\epsilon\alpha_2}{2\omega_2}\xi_1^2 - \frac{i\epsilon}{2\omega_2}\left[\frac{\alpha_{10} z_1}{\omega_1^2-\Omega^2} + \frac{2\alpha_7 z_2}{\omega_2^2-\Omega^2} + \frac{2\alpha_8 z_3}{\omega_3^2-\Omega^2}\right]\bar{\xi}_3 \tag{8.31}$$

$$\dot{\xi}_3 = i\omega_3\xi_3 - \epsilon\mu_3\xi_3 - \frac{i\epsilon}{2\omega_3}\left[\frac{\alpha_{10} z_1}{\omega_1^2-\Omega^2} + \frac{2\alpha_7 z_2}{\omega_2^2-\Omega^2} + \frac{2\alpha_8 z_3}{\omega_3^2-\Omega^2}\right]\bar{\xi}_2 \tag{8.32}$$

$\Omega \approx \omega_3 - \omega_1$

$$\dot{\xi}_1 = i\omega_1\xi_1 - \epsilon\mu_1\xi_1 - \frac{i\epsilon\alpha_2}{\omega_1}\xi_2\bar{\xi}_1 - \frac{i\epsilon}{2\omega_1}\left[\frac{2\alpha_4 \bar{z}_1}{\omega_1^2-\Omega^2} + \frac{\alpha_{10} \bar{z}_2}{\omega_2^2-\Omega^2} + \frac{2\alpha_5 \bar{z}_3}{\omega_3^2-\Omega^2}\right]\xi_3 \tag{8.33}$$

$$\dot{\xi}_2 = i\omega_2\xi_2 - \epsilon\mu_2\xi_2 - \frac{i\epsilon\alpha_2}{2\omega_2}\xi_1^2 \tag{8.34}$$

$$\dot{\xi}_3 = i\omega_3\xi_3 - \epsilon\mu_3\xi_3 - \frac{i\epsilon}{2\omega_3}\left[\frac{2\alpha_4 z_1}{\omega_1^2-\Omega^2} + \frac{\alpha_{10} z_2}{\omega_2^2-\Omega^2} + \frac{2\alpha_5 z_3}{\omega_3^2-\Omega^2}\right]\xi_1 \tag{8.35}$$

$\Omega \approx 2\omega_2$ and $\Omega \approx \omega_3 + \omega_1$

$$\dot{\xi}_1 = i\omega_1\xi_1 - \epsilon\mu_1\xi_1 - \frac{i\epsilon\alpha_2}{\omega_1}\xi_2\bar{\xi}_1 - \frac{i\epsilon}{2\omega_1}\left[\frac{2\alpha_4 z_1}{\omega_1^2 - \Omega^2} + \frac{\alpha_{10} z_2}{\omega_2^2 - \Omega^2} + \frac{2\alpha_5 z_3}{\omega_3^2 - \Omega^2}\right]\bar{\xi}_3 \quad (8.36)$$

$$\dot{\xi}_2 = i\omega_2\xi_2 - \epsilon\mu_2\xi_2 - \frac{i\epsilon\alpha_2}{2\omega_2}\xi_1^2 - \frac{i\epsilon}{\omega_2}\left[\frac{\alpha_3 z_1}{\omega_1^2 - \Omega^2} + \frac{3\alpha_6 z_2}{\omega_2^2 - \Omega^2} + \frac{\alpha_7 z_3}{\omega_3^2 - \Omega^2}\right]\bar{\xi}_2 \quad (8.37)$$

$$\dot{\xi}_3 = i\omega_3\xi_3 - \epsilon\mu_3\xi_3 - \frac{i\epsilon}{2\omega_3}\left[\frac{2\alpha_4 z_1}{\omega_1^2 - \Omega^2} + \frac{\alpha_{10} z_2}{\omega_2^2 - \Omega^2} + \frac{2\alpha_5 z_3}{\omega_3^2 - \Omega^2}\right]\bar{\xi}_1 \quad (8.38)$$

8.1.2
Combination Autoparametric Resonance

We consider the autoparametric-resonance condition $\omega_3 \approx \omega_2 + \omega_1$, which produces the near-resonance terms $\xi_3\bar{\xi}_2$, $\xi_3\bar{\xi}_1$, and $\xi_1\xi_2$ in (8.11)–(8.13), respectively, In the absence of external resonances, the normal form of (8.11)–(8.13) is

$$\dot{\xi}_1 = i\omega_1\xi_1 - \epsilon\mu_1\xi_1 - \frac{i\epsilon\alpha_{10}}{2\omega_1}\xi_3\bar{\xi}_2 \quad (8.39)$$

$$\dot{\xi}_2 = i\omega_2\xi_2 - \epsilon\mu_2\xi_2 - \frac{i\epsilon\alpha_{10}}{2\omega_2}\xi_3\bar{\xi}_1 \quad (8.40)$$

$$\dot{\xi}_3 = i\omega_3\xi_3 - \epsilon\mu_3\xi_3 - \frac{i\epsilon\alpha_{10}}{2\omega_3}\xi_1\xi_2 \quad (8.41)$$

When expressed in the polar coordinates $\xi_n = 1/2 r_n e^{i\beta_n}$, (8.39)–(8.41) become

$$\dot{r}_1 = -\epsilon\mu_1 r_1 + \frac{\epsilon\alpha_{10}}{4\omega_1} r_2 r_3 \sin(\beta_3 - \beta_2 - \beta_1) \quad (8.42)$$

$$\dot{r}_2 = -\epsilon\mu_2 r_2 + \frac{\epsilon\alpha_{10}}{4\omega_2} r_1 r_3 \sin(\beta_3 - \beta_2 - \beta_1) \quad (8.43)$$

$$\dot{r}_3 = -\epsilon\mu_3 r_3 - \frac{\epsilon\alpha_{10}}{4\omega_3} r_1 r_2 \sin(\beta_3 - \beta_2 - \beta_1) \quad (8.44)$$

$$r_1\dot{\beta}_1 = \omega_1 r_1 - \frac{\epsilon\alpha_{10}}{4\omega_1} r_2 r_3 \cos(\beta_3 - \beta_2 - \beta_1) \quad (8.45)$$

$$r_2\dot{\beta}_2 = \omega_2 r_2 - \frac{\epsilon\alpha_{10}}{4\omega_2} r_1 r_3 \cos(\beta_3 - \beta_2 - \beta_1) \quad (8.46)$$

$$r_3\dot{\beta}_3 = \omega_3 r_3 - \frac{\epsilon\alpha_{10}}{4\omega_3} r_1 r_2 \cos(\beta_3 - \beta_2 - \beta_1) \quad (8.47)$$

The normal forms in cases of external resonances can be obtained by adding appropriate terms to (8.39)–(8.41), as in Section 8.1.1.

8.1.3
Simultaneous Two-to-One Autoparametric Resonances

We consider the autoparametric resonances $\omega_3 \approx 2\omega_2$ and $\omega_2 \approx 2\omega_1$, which produce the near-resonance terms $\xi_2\bar{\xi}_1$, ξ_1^2 and $\xi_3\bar{\xi}_2$, and ξ_2^2 in (8.11)–(8.13), respectively. Consequently, in the absence of external resonances, the normal form of (8.11)–(8.13) is

$$\dot{\xi}_1 = i\omega_1\xi_1 - \epsilon\mu_1\xi_1 - \frac{i\epsilon\alpha_2}{\omega_1}\xi_2\bar{\xi}_1 \tag{8.48}$$

$$\dot{\xi}_2 = i\omega_2\xi_2 - \epsilon\mu_2\xi_2 - \frac{i\epsilon}{2\omega_2}\left(\alpha_2\xi_1^2 + 2\alpha_7\xi_3\bar{\xi}_2\right) \tag{8.49}$$

$$\dot{\xi}_3 = i\omega_3\xi_3 - \epsilon\mu_3\xi_3 - \frac{i\epsilon\alpha_7}{2\omega_3}\xi_2^2 \tag{8.50}$$

When expressed in the polar coordinates $\xi_n = 1/2 r_n e^{i\beta_n}$, (8.48)–(8.50) become

$$\dot{r}_1 = -\epsilon\mu_1 r_1 + \frac{\epsilon\alpha_2}{2\omega_1} r_1 r_2 \sin\left(\beta_2 - 2\beta_1\right) \tag{8.51}$$

$$\dot{r}_2 = -\epsilon\mu_2 r_2 - \frac{\epsilon\alpha_2}{4\omega_2} r_1^2 \sin\left(\beta_2 - 2\beta_1\right) + \frac{\epsilon\alpha_7}{2\omega_2} r_2 r_3 \sin\left(\beta_3 - 2\beta_2\right) \tag{8.52}$$

$$\dot{r}_3 = -\epsilon\mu_3 r_3 - \frac{\epsilon\alpha_7}{4\omega_3} r_2^2 \sin\left(\beta_3 - 2\beta_2\right) \tag{8.53}$$

$$\dot{r}_1\dot{\beta}_1 = \omega_1 r_1 - \frac{\epsilon\alpha_2}{2\omega_1} r_1 r_2 \cos\left(\beta_2 - 2\beta_1\right) \tag{8.54}$$

$$\dot{r}_2\dot{\beta}_2 = \omega_2 r_2 - \frac{\epsilon\alpha_2}{4\omega_2} r_1^2 \cos\left(\beta_2 - 2\beta_1\right) - \frac{\epsilon\alpha_7}{2\omega_2} r_2 r_3 \cos\left(\beta_3 - 2\beta_2\right) \tag{8.55}$$

$$\dot{r}_3\dot{\beta}_3 = \omega_3 r_3 - \frac{\epsilon\alpha_7}{4\omega_3} r_2^2 \cos\left(\beta_3 - 2\beta_2\right) \tag{8.56}$$

This form is in full agreement with that obtained by Nayfeh, Asrar, and Nayfeh (1992) by using the method of multiple scales.

Again, the normal forms in the presence of external resonances can be obtained by adding appropriate terms to (8.48)–(8.50), as in Section 8.1.1.

8.1.4
Primary Resonances

To treat this case, we scale the excitation amplitudes f_n and hence the z_n at $O(\epsilon)$ so that the resonance terms produced by the excitation appear at the same order as those produced by the nonlinearity. Thus, we modify (8.7) into

$$\begin{aligned}\dot{\zeta}_1 = {} & i\omega_1\zeta_1 - \epsilon\mu_1\left(\zeta_1 - \bar{\zeta}_1\right) - \frac{i\epsilon}{2\omega_1}\left(z_1 + \bar{z}_1\right) - \frac{i\epsilon}{2\omega_1}\Big\{3\alpha_1\left(\zeta_1 + \bar{\zeta}_1\right)^2 \\ & + 2\alpha_2\left(\zeta_1 + \bar{\zeta}_1\right)\left(\zeta_2 + \bar{\zeta}_2\right) + \alpha_3\left(\zeta_2 + \bar{\zeta}_2\right)^2 + 2\alpha_4\left(\zeta_1 + \bar{\zeta}_1\right)\left(\zeta_3 + \bar{\zeta}_3\right) \\ & + \alpha_5\left(\zeta_3 + \bar{\zeta}_3\right)^2 + \alpha_{10}\left(\zeta_2 + \bar{\zeta}_2\right)\left(\zeta_3 + \bar{\zeta}_3\right)\Big\}\end{aligned} \tag{8.57}$$

$$\dot{\zeta}_2 = i\omega_2\zeta_2 - \epsilon\mu_2\left(\zeta_2 - \bar{\zeta}_2\right) - \frac{i\epsilon}{2\omega_2}\left(z_2 + \bar{z}_2\right) - \frac{i\epsilon}{2\omega_2}\left\{\alpha_2\left(\zeta_1 + \bar{\zeta}_1\right)^2 + 2\alpha_3\left(\zeta_1 + \bar{\zeta}_1\right)\left(\zeta_2 + \bar{\zeta}_2\right) + 3\alpha_6\left(\zeta_2 + \bar{\zeta}_2\right)^2 + 2\alpha_7\left(\zeta_2 + \bar{\zeta}_2\right)\left(\zeta_3 + \bar{\zeta}_3\right) + \alpha_8\left(\zeta_3 + \bar{\zeta}_3\right)^2 + \alpha_{10}\left(\zeta_1 + \bar{\zeta}_1\right)\left(\zeta_3 + \bar{\zeta}_3\right)\right\} \tag{8.58}$$

$$\dot{\zeta}_3 = i\omega_3\zeta_3 - \epsilon\mu_3\left(\zeta_3 - \bar{\zeta}_3\right) - \frac{i\epsilon}{2\omega_3}\left(z_3 + \bar{z}_3\right) - \frac{i\epsilon}{2\omega_3}\left\{\alpha_4\left(\zeta_1 + \bar{\zeta}_1\right)^2 + 2\alpha_5\left(\zeta_1 + \bar{\zeta}_1\right)\left(\zeta_3 + \bar{\zeta}_3\right) + \alpha_7\left(\zeta_2 + \bar{\zeta}_2\right)^2 + 2\alpha_8\left(\zeta_2 + \bar{\zeta}_2\right)\left(\zeta_3 + \bar{\zeta}_3\right) + 3\alpha_9\left(\zeta_3 + \bar{\zeta}_3\right)^2 + \alpha_{10}\left(\zeta_1 + \bar{\zeta}_1\right)\left(\zeta_2 + \bar{\zeta}_2\right)\right\} \tag{8.59}$$

To simplify (8.57)–(8.59), we introduce the near-identity transformation

$$\zeta_m = \xi_m + \epsilon h_m\left(\xi_n, \bar{\xi}_n, z_n, \bar{z}_n\right) \tag{8.60}$$

and choose the h_m to eliminate the nonresonance terms. The resulting normal forms depend on the resonance conditions being considered. As discussed in the preceding section, internal or autoparametric resonances occur when $\omega_n \approx 2\omega_m$ and/or $\omega_s \approx \omega_n \pm \omega_m$. Primary resonances occur when $\Omega \approx \omega_m$. Consequently, the normal forms of (8.57)–(8.59) for a few of these resonances are:

$\omega_2 \approx 2\omega_1$ and $\Omega \approx \omega_n$

$$\dot{\xi}_1 = i\omega_1\xi_1 - \epsilon\mu_1\xi_1 - \frac{i\epsilon\alpha_2}{\omega_1}\xi_2\bar{\xi}_1 - \frac{i\epsilon}{2\omega_1}\delta_{1n}z_1 \tag{8.61}$$

$$\dot{\xi}_2 = i\omega_2\xi_2 - \epsilon\mu_2\xi_2 - \frac{i\epsilon\alpha_2}{2\omega_2}\xi_1^2 - \frac{i\epsilon}{2\omega_2}\delta_{2n}z_2 \tag{8.62}$$

$$\dot{\xi}_3 = i\omega_3\xi_3 - \epsilon\mu_3\xi_3 - \frac{i\epsilon}{2\omega_3}\delta_{3n}z_3 \tag{8.63}$$

where δ_{mn} is the Kronecker delta.

$\omega_3 \approx \omega_2 + \omega_1$ and $\Omega \approx \omega_n$

$$\dot{\xi}_1 = i\omega_1\xi_1 - \epsilon\mu_1\xi_1 - \frac{i\epsilon\alpha_{10}}{2\omega_1}\xi_3\bar{\xi}_2 - \frac{i\epsilon}{2\omega_1}\delta_{1n}z_1 \tag{8.64}$$

$$\dot{\xi}_2 = i\omega_2\xi_2 - \epsilon\mu_2\xi_2 - \frac{i\epsilon\alpha_{10}}{2\omega_2}\xi_3\bar{\xi}_1 - \frac{i\epsilon}{2\omega_2}\delta_{2n}z_2 \tag{8.65}$$

$$\dot{\xi}_3 = i\omega_3\xi_3 - \epsilon\mu_3\xi_3 - \frac{i\epsilon\alpha_{10}}{2\omega_3}\xi_1\xi_2 - \frac{i\epsilon}{2\omega_3}\delta_{3n}z_3 \tag{8.66}$$

$\omega_3 \approx 2\omega_2, \omega_2 \approx 2\omega_1$, and $\Omega \approx \omega_n$

$$\dot{\xi}_1 = i\omega_1\xi_1 - \epsilon\mu_1\xi_1 - \frac{i\epsilon\alpha_2}{\omega_1}\xi_2\bar{\xi}_1 - \frac{i\epsilon}{2\omega_1}\delta_{1n}z_1 \tag{8.67}$$

$$\dot{\xi}_2 = i\omega_2\xi_2 - \epsilon\mu_2\xi_2 - \frac{i\epsilon}{2\omega_2}\left(\alpha_2\xi_1^2 + 2\alpha_7\xi_3\bar{\xi}_2\right) - \frac{i\epsilon}{2\omega_2}\delta_{2n}z_2 \tag{8.68}$$

$$\dot{\xi}_3 = i\omega_3\xi_3 - \epsilon\mu_3\xi_3 - \frac{i\epsilon\alpha_7}{2\omega_3}\xi_2^2 - \frac{i\epsilon}{2\omega_3}\delta_{3n}z_3 \tag{8.69}$$

$\omega_3 \approx \omega_2 + \omega_1, \omega_2 \approx 2\omega_1$, and $\Omega \approx \omega_n$

$$\dot{\xi}_1 = i\omega_1\xi_1 - \epsilon\mu_1\xi_1 - \frac{i\epsilon}{2\omega_1}\left(2\alpha_2\xi_2\bar{\xi}_1 + \alpha_{10}\xi_3\bar{\xi}_2\right) - \frac{i\epsilon}{2\omega_1}\delta_{1n}z_1 \tag{8.70}$$

$$\dot{\xi}_2 = i\omega_2\xi_2 - \epsilon\mu_2\xi_2 - \frac{i\epsilon}{2\omega_2}\left(\alpha_2\xi_1^2 + \alpha_{10}\xi_3\bar{\xi}_1\right) - \frac{i\epsilon}{2\omega_2}\delta_{2n}z_2 \tag{8.71}$$

$$\dot{\xi}_3 = i\omega_3\xi_3 - \epsilon\mu_3\xi_3 - \frac{i\epsilon\alpha_{10}}{2\omega_3}\xi_1\xi_2 - \frac{i\epsilon}{2\omega_3}\delta_{3n}z_3 \tag{8.72}$$

8.2 Gyroscopic Systems

As in Section 7.4, for simplicity, we consider a two-degree-of-freedom system to illustrate the method of solution. Specifically, we consider

$$\ddot{u}_1 + \lambda_1\dot{u}_2 + \alpha_1 u_1 = \epsilon\left(3\delta_1 u_1^2 + 2\delta_2 u_1 u_2 + \delta_3 u_2^2\right) + 2F_1\cos\left(\Omega t + \tau_1\right) \tag{8.73}$$

$$\ddot{u}_2 - \lambda_2\dot{u}_1 + \alpha_2 u_2 = \epsilon\left(\delta_2 u_1^2 + 2\delta_3 u_1 u_2 + 3\delta_4 u_2^2\right) + 2F_2\cos\left(\Omega t + \tau_2\right) \tag{8.74}$$

Using the transformation (7.144)–(7.147) and following steps similar to those used in Section 7.4, we cast (8.73) and (8.74) in the complex-valued form

$$\begin{aligned}\dot{\zeta}_1 = {} & i\omega_1\zeta_1 + \frac{i\omega_1(\alpha_1-\omega_2^2)}{2\alpha_1(\omega_2^2-\omega_1^2)}\left(z_1+\bar{z}_1\right) - \frac{\lambda_1}{2(\omega_2^2-\omega_1^2)}\left(z_2+\bar{z}_2\right) \\ & + \epsilon\chi_{11}\left(\zeta_1+\bar{\zeta}_1+\zeta_2+\bar{\zeta}_2\right)^2 + \epsilon\chi_{12}\left(\zeta_1+\bar{\zeta}_1+\zeta_2+\bar{\zeta}_2\right) \\ & \bullet\left[\frac{i(\alpha_1-\omega_1^2)}{\lambda_1\omega_1}\left(\zeta_1-\bar{\zeta}_1\right) + \frac{i(\alpha_1-\omega_2^2)}{\lambda_1\omega_2}\left(\zeta_2-\bar{\zeta}_2\right)\right] \\ & + \epsilon\chi_{13}\left[\frac{i(\alpha_1-\omega_1^2)}{\lambda_1\omega_1}\left(\zeta_1-\bar{\zeta}_1\right) + \frac{i(\alpha_1-\omega_2^2)}{\lambda_1\omega_2}\left(\zeta_2-\bar{\zeta}_2\right)\right]^2\end{aligned} \tag{8.75}$$

$$\begin{aligned}\dot{\zeta}_2 &= i\omega_2\zeta_2 - \frac{i\omega_2(\alpha_1-\omega_1^2)}{2\alpha_1(\omega_2^2-\omega_1^2)}(z_1+\bar{z}_1) + \frac{\lambda_1}{2(\omega_2^2-\omega_1^2)}(z_2+\bar{z}_2) \\ &\quad + \epsilon\chi_{21}\left(\zeta_1+\bar{\zeta}_1+\zeta_2+\bar{\zeta}_2\right)^2 + \epsilon\chi_{22}\left(\zeta_1+\bar{\zeta}_1+\zeta_2+\bar{\zeta}_2\right) \\ &\quad \bullet \left[\frac{i(\alpha_1-\omega_1^2)}{\lambda_1\omega_1}\left(\zeta_1-\bar{\zeta}_1\right) + \frac{i(\alpha_1-\omega_2^2)}{\lambda_1\omega_2}\left(\zeta_2-\bar{\zeta}_2\right)\right] \\ &\quad + \epsilon\chi_{23}\left[\frac{i(\alpha_1-\omega_1^2)}{\lambda_1\omega_1}\left(\zeta_1-\bar{\zeta}_1\right) + \frac{i(\alpha_1-\omega_2^2)}{\lambda_1\omega_2}\left(\zeta_2-\bar{\zeta}_2\right)\right]^2\end{aligned} \tag{8.76}$$

on account of (8.6), where

$$\chi_{11} = \frac{3i\delta_1\omega_1(\alpha_1-\omega_2^2)-\delta_2\alpha_1\lambda_1}{2\alpha_1(\omega_2^2-\omega_1^2)}, \quad \chi_{12} = \frac{i\delta_2\omega_1(\alpha_1-\omega_2^2)-\delta_3\alpha_1\lambda_1}{\alpha_1(\omega_2^2-\omega_1^2)}$$

$$\chi_{13} = \frac{i\delta_3\omega_1(\alpha_1-\omega_2^2)-3\delta_4\alpha_1\lambda_1}{2\alpha_1(\omega_2^2-\omega_1^2)}, \quad \chi_{21} = \frac{\delta_2\alpha_1\lambda_1-3i\delta_1\omega_2(\alpha_1-\omega_1^2)}{2\alpha_1(\omega_2^2-\omega_1^2)}$$

$$\chi_{22} = \frac{\delta_3\alpha_1\lambda_1-i\delta_2\omega_2(\alpha_1-\omega_1^2)}{\alpha_1(\omega_2^2-\omega_1^2)}, \quad \chi_{33} = \frac{3\delta_4\alpha_1\lambda_1-i\delta_3\omega_2(\alpha_1-\omega_1^2)}{2\alpha_1(\omega_2^2-\omega_1^2)}$$

As discussed in Section 8.1, the excitation may produce one of two possible types of resonances: primary and secondary resonances. Primary resonances are treated in the next section, and secondary resonances are treated in Section 8.2.2.

8.2.1
Primary Resonances

As discussed in Section 8.1.4, we scale the f_n and hence the z_n at $O(\epsilon)$ so that the resonance terms produced by the excitation appear at the same order as those produced by the nonlinearity.

To simplify (8.75) and (8.76), we introduce the near-identity transformation (8.60) and choose the h_m to eliminate the nonresonance terms. Resonance terms occur when $\omega_2 \approx 2\omega_1$ (i.e., when there is a two-to-one autoparametric resonance) and $\Omega \approx \omega_n$ (i.e., when there is a primary resonance of the nth mode). When $\omega_2 \approx 2\omega_1$ and $\Omega \approx \omega_n$, the normal form of (8.75) and (8.76) is

$$\dot{\xi}_1 = i\omega_1\xi_1 + \epsilon\Lambda_1\xi_2\bar{\xi}_1 + \epsilon\frac{i\omega_1(\alpha_1-\omega_2^2)z_1-\lambda_1\alpha_1 z_2}{2\alpha_1(\omega_2^2-\omega_1^2)}\delta_{1n} \tag{8.77}$$

$$\dot{\xi}_2 = i\omega_2\xi_2 + \epsilon\Lambda_2\xi_1^2 - \epsilon\frac{i\omega_2(\alpha_1-\omega_1^2)z_1-\lambda\alpha_1 z_2}{2\alpha_1(\omega_2^2-\omega_1^2)}\delta_{2n} \tag{8.78}$$

where

$$\Lambda_1 = 2\chi_{11} - \frac{i\chi_{12}(\omega_2-\omega_1)(\alpha_1+\omega_1\omega_2)}{\lambda_1\omega_1\omega_2} - \frac{2\chi_{13}\alpha_1\lambda_2}{\lambda_1\omega_1\omega_2} \tag{8.79}$$

$$\Lambda_2 = \chi_{21} + \frac{i\chi_{22}(\alpha_1-\omega_1^2)}{\lambda_1\omega_1} - \frac{\chi_{23}(\alpha_1-\omega_1^2)^2}{\lambda_1^2\omega_1^2} \tag{8.80}$$

8.2.2
Secondary Resonances

To treat this case, we first introduce the linear transformation (8.8) and choose $\varDelta_{m1}$ and $\varDelta_{m2}$ to eliminate z_n and $\bar{z}_n$ from (8.75) and (8.76); that is,

$$\begin{aligned}\zeta_1 = \eta_1 &- \frac{\omega_1(\alpha_1-\omega_2^2)}{2\alpha_1(\omega_2^2-\omega_1^2)}\left(\frac{z_1}{\omega_1-\Omega}+\frac{\bar{z}_1}{\omega_1+\Omega}\right)\\ &- \frac{i\lambda_1}{2(\omega_2^2-\omega_1^2)}\left(\frac{z_2}{\omega_1-\Omega}+\frac{\bar{z}_2}{\omega_1+\Omega}\right)\end{aligned} \tag{8.81}$$

$$\begin{aligned}\zeta_2 = \eta_2 &+ \frac{\omega_2(\alpha_1-\omega_1^2)}{2\alpha_1(\omega_2^2-\omega_1^2)}\left(\frac{z_1}{\omega_2-\Omega}+\frac{\bar{z}_1}{\omega_2+\Omega}\right)\\ &+ \frac{i\lambda_1}{2(\omega_2^2-\omega_1^2)}\left(\frac{z_2}{\omega_2-\Omega}+\frac{\bar{z}_2}{\omega_2+\Omega}\right)\end{aligned} \tag{8.82}$$

Hence, (8.75) and (8.76) become

$$\begin{aligned}\dot{\eta}_1 = i\omega_1\eta_1 &+ \epsilon\chi_{11}\left[\eta_1+\bar{\eta}_1+\eta_2+\bar{\eta}_2+b_1(z_1+\bar{z}_1)+b_2(z_2-\bar{z}_2)\right]^2\\ &+\epsilon\chi_{12}\left[\eta_1+\bar{\eta}_1+\eta_2+\bar{\eta}_2+b_1(z_1+\bar{z}_1)+b_2(z_2-\bar{z}_2)\right]\\ &\bullet\left[\frac{i(\alpha_1-\omega_1^2)}{\lambda_1\omega_1}(\eta_1-\bar{\eta}_1)+\frac{i(\alpha_1-\omega_2^2)}{\lambda_1\omega_2}(\eta_2-\bar{\eta}_2)+b_3(z_1-\bar{z}_1)\right.\\ &\left.+b_4(z_2+\bar{z}_2)\right]+\epsilon\chi_{13}\left[\frac{i(\alpha_1-\omega_1^2)}{\lambda_2\omega_1}(\eta_1-\bar{\eta}_1)\right.\\ &\left.+\frac{i(\alpha_1-\omega_2^2)}{\lambda_1\omega_2}(\eta_2-\bar{\eta}_2)+b_3(z_1-\bar{z}_1)+b_4(z_2+\bar{z}_2)\right]^2\end{aligned} \tag{8.83}$$

$$\begin{aligned}\dot{\eta}_2 = i\omega_2\eta_2 &+ \epsilon\chi_{21}\left[\eta_1+\bar{\eta}_1+\eta_2+\bar{\eta}_2+b_1(z_1+\bar{z}_1)+b_2(z_2-\bar{z}_2)\right]^2\\ &+\epsilon\chi_{22}\left[\eta_1+\bar{\eta}_1+\eta_2+\bar{\eta}_2+b_1(z_1+\bar{z}_1)+b_2(z_2-\bar{z}_2)\right]\\ &\bullet\left[\frac{i(\alpha_1-\omega_1^2)}{\lambda_1\omega_1}(\eta_1-\bar{\eta}_1)+\frac{i(\alpha_1-\omega_2^2)}{\lambda_1\omega_2}(\eta_2-\bar{\eta}_2)+b_3(z_1-\bar{z}_1)\right.\\ &\left.+b_4(z_2+\bar{z}_2)\right]+\epsilon\chi_{23}\left[\frac{i(\alpha_1-\omega_1^2)}{\lambda_1\omega_1}(\eta_1-\bar{\eta}_1)\right.\\ &\left.+\frac{i(\alpha_1-\omega_2^2)}{\lambda_1\omega_2}(\eta_2-\bar{\eta}_2)+b_3(z_1-\bar{z}_1)+b_4(z_2+\bar{z}_2)\right]^2\end{aligned} \tag{8.84}$$

where

$$(b_1, b_2, b_3, b_4) = \frac{(\alpha_2-\Omega^2, -i\lambda_1\Omega, i\lambda_2\Omega, \alpha_1-\Omega^2)}{(\omega_1^2-\Omega^2)(\omega_2^2-\Omega^2)} \tag{8.85}$$

It follows from (8.83) and (8.84) that, to this order, only a two-to-one autoparametric resonance is possible and that the excitation can produce resonances corresponding to

- $\Omega \approx 2\omega_m$: Subharmonic resonance of order one-half of the mth mode,
- $\Omega \approx 1/2\omega_m$: Superharmonic resonance of order two of the mth mode,
- $\Omega \approx \omega_2 \pm \omega_1$: Combination resonance of the additive or difference types.

Next, we present the normal forms for several of the external resonances for the case of two-to-one autoparametric resonance. Because (8.83) and (8.84) were derived under the assumption that Ω is away from ω_1 and ω_2, the resonance conditions $\Omega \approx 2\omega_1$, $\Omega \approx 1/2\omega_2$, and $\Omega \approx \omega_2 - \omega_1$ must be excluded when considering the two-to-one autoparametric resonance.

$\Omega \approx 2\omega_2$

$$\dot{\xi}_1 = i\omega_1\xi_1 + \epsilon\Lambda_1\xi_2\bar{\xi}_1 \tag{8.86}$$

$$\dot{\xi}_2 = i\omega_2\xi_2 + \epsilon\Lambda_2\xi_1^2 + \epsilon\Gamma\bar{\xi}_2 \tag{8.87}$$

where Λ_1 and Λ_2 are defined in (8.79) and (8.80) and

$$\begin{aligned}\Gamma &= 2\chi_{21}\,(b_1z_1 + b_2z_2) - 2i\chi_{23}\frac{(\alpha_1 - \omega_2^2)}{\lambda_1\omega_2}\,(b_3z_1 + b_4z_2)\\ &\quad + \chi_{22}\left[b_3z_1 + b_4z_2 - \frac{i(\alpha_1 - \omega_2^2)}{\lambda_1\omega_2}\,(b_1z_1 + b_2z_2)\right]\end{aligned} \tag{8.88}$$

$\Omega \approx 1/2\omega_1$

$$\dot{\xi}_1 = i\omega_1\xi_1 + \epsilon\Lambda_1\xi_2\bar{\xi}_1 + \epsilon\Gamma \tag{8.89}$$

$$\dot{\xi}_2 = i\omega_2\xi_2 + \epsilon\Lambda_2\xi_1^2 \tag{8.90}$$

where Λ_1 and Λ_2 are defined in (8.79) and (8.80) and

$$\begin{aligned}\Gamma &= \chi_{11}\,(b_1z_1 + b_2z_2)^2 + \chi_{12}\,(b_1z_1 + b_2z_2)\,(b_3z_1 + b_4z_2)\\ &\quad + \chi_{13}\,(b_3z_1 + b_4z_2)^2\end{aligned} \tag{8.91}$$

$\Omega \approx \omega_2 + \omega_1$

$$\dot{\xi}_1 = i\omega_1\xi_1 + \epsilon\Lambda_1\xi_2\bar{\xi}_1 + \epsilon\Gamma_1\bar{\xi}_2 \tag{8.92}$$

$$\dot{\xi}_2 = i\omega_2\xi_2 + \epsilon\Lambda_2\xi_1^2 + \epsilon\Gamma_2\bar{\xi}_1 \tag{8.93}$$

where Λ_1 and Λ_2 are defined in (8.79) and (8.80) and

$$\begin{aligned}\Gamma_1 &= 2\chi_{11}\,(b_1z_1 + b_2z_2) - 2\chi_{13}\frac{i\,(\alpha_1 - \omega_2^2)}{\lambda_1\omega_2}\,(b_3z_1 + b_4z_2)\\ &\quad + \chi_{12}\left[b_3z_1 + b_4z_2 - \frac{i\,(\alpha_1 - \omega_2^2)}{\lambda_1\omega_2}\,(b_1z_1 + b_2z_2)\right]\end{aligned} \tag{8.94}$$

$$\Gamma_2 = 2\chi_{21}\left(b_1 z_1 + b_2 z_2\right) - 2\chi_{23}\frac{i\left(\alpha_1 - \omega_1^2\right)}{\lambda_1\omega_1}\left(b_3 z_1 + b_4 z_2\right) + \chi_{22}\left[b_3 z_1 + b_4 z_2 - \frac{i\left(\alpha_1 - \omega_1^2\right)}{\lambda_1\omega_1}\left(b_1 z_1 + b_2 z_2\right)\right] \tag{8.95}$$

8.3
Two Linearly Coupled Oscillators

In this section, we consider the two linearly coupled oscillators

$$\ddot{u}_1 + k_{11}u_1 + k_{12}u_2 = -2\epsilon d_1\dot{u}_1 + \epsilon\left(\alpha_{11}u_1^2 + \alpha_{12}u_2^2 + \alpha_{13}u_1u_2\right) \tag{8.96}$$

$$\ddot{u}_2 + k_{21}u_1 + k_{22}u_2 = -2\epsilon d_2\dot{u}_2 + \epsilon\left(\alpha_{21}u_1^2 + \alpha_{22}u_2^2 + \alpha_{23}u_1u_2\right) \tag{8.97}$$

To determine the normal form of (8.96) and (8.97), we use two approaches. First, we use these equations as they are. Second, we first transform them so that their linear parts are in normal-mode form, as in Section 8.1.

The solution of the linearly undamped equations (8.96) and (8.97) can be expressed as

$$u_1 = A_1 e^{i\omega_1 t} + A_2 e^{i\omega_2 t} + \text{cc} \tag{8.98}$$

$$u_2 = \Gamma_1 A_1 e^{i\omega_2 t} + \Gamma_2 A_2 e^{i\omega_2 t} + \text{cc} \tag{8.99}$$

when $\omega_1 \neq \omega_2$, where

$$\omega^4 - \left(k_{11} + k_{22}\right)\omega^2 + k_{11}k_{22} - k_{12}k_{21} = 0$$
$$\Gamma_i = -\frac{k_{11} - \omega_1^2}{k_{12}} = -\frac{k_{21}}{k_{22} - \omega_i^2} \quad \text{for } i = 1 \text{ and } 2 \tag{8.100}$$

Using (8.98) and (8.99), we express (8.96) and (8.97) as a system of two first-order complex-valued equations by using the transformation

$$u_1 = \zeta_1 + \bar{\zeta}_1 + \zeta_2 + \bar{\zeta}_2 \tag{8.101}$$

$$u_2 = \Gamma_1\left(\zeta_1 + \bar{\zeta}_1\right) + \Gamma_2\left(\zeta_2 + \bar{\zeta}_2\right) \tag{8.102}$$

$$\dot{u}_1 = i\omega_1\left(\zeta_1 - \bar{\zeta}_1\right) + i\omega_2\left(\zeta_2 - \bar{\zeta}_2\right) \tag{8.103}$$

$$\dot{u}_2 = i\omega_1\Gamma_1\left(\zeta_1 - \bar{\zeta}_1\right) + i\omega_2\Gamma_2\left(\zeta_1 - \bar{\zeta}_2\right) \tag{8.104}$$

Solving (8.101)–(8.104), we have

$$\zeta_1 = \frac{1}{2(\Gamma_2 - \Gamma_1)} \left[\Gamma_2 u_1 - u_2 + \frac{i}{\omega_1} (\dot{u}_2 - \Gamma_2 \dot{u}_1) \right] \tag{8.105}$$

$$\zeta_2 = \frac{1}{2(\Gamma_2 - \Gamma_1)} \left[u_2 - \Gamma_1 u_1 + \frac{i}{\omega_2} (\Gamma_1 \dot{u}_1 - \dot{u}_2) \right] \tag{8.106}$$

Differentiating (8.105) and (8.106) with respect to t, we obtain

$$\dot{\zeta}_1 = \frac{1}{2(\Gamma_2 - \Gamma_1)} \left[\Gamma_2 \dot{u}_1 - \dot{u}_2 + \frac{i}{\omega_1} (\ddot{u}_2 - \Gamma_2 \ddot{u}_1) \right] \tag{8.107}$$

$$\dot{\zeta}_2 = \frac{1}{2(\Gamma_2 - \Gamma_1)} \left[\dot{u}_2 - \Gamma_1 \dot{u}_1 + \frac{i}{\omega_2} (\Gamma_1 \ddot{u}_1 - \ddot{u}_2) \right] \tag{8.108}$$

Eliminating $\ddot{u}_1$ and $\ddot{u}_2$ from (8.107) and (8.108) by using (8.96) and (8.97), we obtain

$$\begin{aligned}\dot{\zeta}_1 &= i\omega_1\zeta_1 + \frac{\epsilon(d_2\Gamma_1 - d_1\Gamma_2)}{\Gamma_2 - \Gamma_1}\left(\zeta_1 - \bar{\zeta}_1\right) + \frac{\epsilon\omega_2\Gamma_2(d_2 - d_1)}{\omega_1(\Gamma_2 - \Gamma_1)}\left(\zeta_2 - \bar{\zeta}_2\right) \\ &+ \frac{i\epsilon}{2\omega_1(\Gamma_2 - \Gamma_1)}\Big\{(\alpha_{21} - \Gamma_2\alpha_{11})\left[\zeta_1 + \bar{\zeta}_1 + \zeta_2 + \bar{\zeta}_2\right]^2 \\ &+ (\alpha_{22} - \Gamma_2\alpha_{12})\left[\Gamma_1\left(\zeta_1 + \bar{\zeta}_1\right) + \Gamma_2\left(\zeta_2 + \bar{\zeta}_2\right)\right]^2 \\ &+ (\alpha_{23} - \Gamma_2\alpha_{13})\left[\zeta_1 + \bar{\zeta}_1 + \zeta_2 + \bar{\zeta}_2\right]\left[\Gamma_1\left(\zeta_1 + \bar{\zeta}_1\right) + \Gamma_2\left(\zeta_2 + \bar{\zeta}_2\right)\right]\Big\}\end{aligned} \tag{8.109}$$

$$\begin{aligned}\dot{\zeta}_2 &= i\omega_2\zeta_2 + \frac{\epsilon\omega_1\Gamma_1(d_1 - d_2)}{\omega_2(\Gamma_2 - \Gamma_1)}\left(\zeta_1 - \bar{\zeta}_1\right) + \frac{\epsilon(d_1\Gamma_1 - d_2\Gamma_2)}{\Gamma_2 - \Gamma_1}\left(\zeta_2 - \bar{\zeta}_2\right) \\ &+ \frac{i\epsilon}{2\omega_2(\Gamma_2 - \Gamma_1)}\Big\{(\Gamma_1\alpha_{11} - \alpha_{21})\left[\zeta_1 + \bar{\zeta}_1 + \zeta_2 + \bar{\zeta}_2\right]^2 \\ &+ (\Gamma_1\alpha_{12} - \alpha_{22})\left[\Gamma_1\left(\zeta_1 + \bar{\zeta}_1\right) + \Gamma_2\left(\zeta_2 + \bar{\zeta}_2\right)\right]^2 \\ &+ (\Gamma_1\alpha_{13} - \alpha_{23})\left[\zeta_1 + \bar{\zeta}_1 + \zeta_2 + \bar{\zeta}_2\right]\left[\Gamma_1\left(\zeta_1 + \bar{\zeta}_1\right) + \Gamma_2\left(\zeta_2 + \bar{\zeta}_2\right)\right]\Big\}\end{aligned} \tag{8.110}$$

To simplify (8.109) and (8.110), we introduce the near-identity transformation

$$\zeta_m = \xi_m + \epsilon h_m\left(\xi_1, \bar{\xi}_1, \xi_2, \bar{\xi}_2\right) \tag{8.111}$$

and choose the h_m to eliminate the nonresonance terms. There are two cases: two-to-one autoparametric resonance (i.e., $\omega_2 \approx 2\omega_1$) and no autoparametric resonance. Next, we present the normal forms for these cases starting with the second.

No Autoparametric Resonance

$$\dot{\xi}_1 = i\omega_1\xi_1 - \epsilon\mu_1\xi_1 \tag{8.112}$$

$$\dot{\xi}_2 = i\omega_2\xi_2 - \epsilon\mu_2\xi_2 \tag{8.113}$$

where

$$\mu_1 = \frac{d_1\Gamma_2 - d_2\Gamma_1}{\Gamma_2 - \Gamma_1} = \frac{d_2(k_{11} - \omega_1^2) - d_1(k_{11} - \omega_2^2)}{\omega_2^2 - \omega_1^2} \tag{8.114}$$

$$\mu_2 = \frac{d_2\Gamma_2 - d_1\Gamma_1}{\Gamma_2 - \Gamma_1} = \frac{d_2(k_{11} - \omega_2^2) - d_1(k_{11} - \omega_1^2)}{\omega_1^2 - \omega_2^2} \tag{8.115}$$

Two-to-One Autoparametric Resonance

$$\dot{\xi}_1 = i\omega_1\xi_1 - \epsilon\mu_1\xi_1 + i\epsilon\Lambda_1\xi_2\bar{\xi}_1 \tag{8.116}$$

$$\dot{\xi}_2 = i\omega_2\xi_2 - \epsilon\mu_2\xi_2 + i\epsilon\Lambda_2\xi_1^2 \tag{8.117}$$

where

$$\Lambda_1 = \frac{1}{2\omega_1(\Gamma_2 - \Gamma_1)}[2(\alpha_{21} - \Gamma_2\alpha_{11}) + 2\Gamma_1\Gamma_2(\alpha_{22} - \Gamma_2\alpha_{12}) + (\Gamma_1 + \Gamma_2)(\alpha_{23} - \Gamma_2\alpha_{13})] \tag{8.118}$$

$$\Lambda_2 = \frac{1}{2\omega_2(\Gamma_2 - \Gamma_1)}\left[\Gamma_1\alpha_{11} - \alpha_{21} + \Gamma_1^2(\Gamma_1\alpha_{12} - \alpha_{22}) + \Gamma_1(\Gamma_1\alpha_{13} - \alpha_{23})\right] \tag{8.119}$$

Next, we first transform the linear parts of (8.96) and (8.97) into a normal-mode form. To accomplish this, we note from (8.98)–(8.100) that the eigenvectors corresponding to the eigenvalues ω_1 and ω_2 are $[1\ \Gamma_1]^T$ and $[1\ \Gamma_2]^T$. Hence, we introduce the transformation

$$u = Pv = \begin{bmatrix} 1 & 1 \\ \Gamma_1 & \Gamma_2 \end{bmatrix} v \tag{8.120}$$

into (8.96) and (8.97) and obtain

$$\begin{bmatrix} 1 & 1 \\ \Gamma_1 & \Gamma_2 \end{bmatrix}\begin{bmatrix} \ddot{v}_1 \\ \ddot{v}_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ \Gamma_1 & \Gamma_2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$
$$= -2\epsilon\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ \Gamma_1 & \Gamma_2 \end{bmatrix}\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix}$$
$$+ \epsilon\begin{bmatrix} \alpha_{11}(v_1 + v_2)^2 + \alpha_{12}(\Gamma_1 v_1 + \Gamma_2 v_2)^2 + \alpha_{13}(v_1 + v_2)(\Gamma_1 v_1 + \Gamma_2 v_2) \\ \alpha_{21}(v_1 + v_2)^2 + \alpha_{22}(\Gamma_1 v_1 + \Gamma_2 v_2)^2 + \alpha_{23}(v_1 + v_2)(\Gamma_1 v_1 + \Gamma_2 v_2) \end{bmatrix} \tag{8.121}$$

Multiplying (8.121) from the left with

$$P^{-1} = \frac{1}{\Gamma_2 - \Gamma_1}\begin{bmatrix} \Gamma_2 & -1 \\ -\Gamma_1 & 1 \end{bmatrix} \tag{8.122}$$

and after some algebraic manipulations, we obtain

$$\begin{bmatrix} \ddot{v}_1 \\ \ddot{v}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -2\epsilon \begin{bmatrix} \mu_1 & \mu_3 \\ \mu_4 & \mu_2 \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix}$$
$$+ \frac{\epsilon}{\Gamma_2 - \Gamma_1} \begin{bmatrix} (\Gamma_2 \alpha_{11} - \alpha_{21})(v_1 + v_2)^2 + (\Gamma_2 \alpha_{12} - \alpha_{22})(\Gamma_1 v_1 + \Gamma_2 v_2)^2 \\ + (\Gamma_2 \alpha_{13} - \alpha_{23})(v_1 + v_2)(\Gamma_1 v_1 + \Gamma_2 v_2) \\ (\alpha_{21} - \Gamma_1 \alpha_{11})(v_1 + v_2)^2 + (\alpha_{22} - \Gamma_1 \alpha_{12})(\Gamma_1 v_1 + \Gamma_2 v_2)^2 \\ + (\alpha_{23} - \Gamma_1 \alpha_{13})(v_1 + v_2)(\Gamma_1 v_1 + \Gamma_2 v_2) \end{bmatrix} \quad (8.123)$$

where μ_1 and μ_2 are defined in (8.114) and (8.115) and

$$\mu_3 = \frac{\Gamma_2(d_1 - d_2)}{\Gamma_2 - \Gamma_1} \quad \text{and} \quad \mu_4 = \frac{\Gamma_1(d_2 - d_1)}{\Gamma_2 - \Gamma_1} \quad (8.124)$$

Next, we transform (8.123) into two first-order complex-valued equations by letting

$$v_1 = \zeta_1 + \bar{\zeta}_1, \quad \dot{v}_1 = i\omega_1 \left(\zeta_1 - \bar{\zeta}_1\right) \quad (8.125)$$

$$v_2 = \zeta_2 + \bar{\zeta}_2, \quad \dot{v}_2 = i\omega_2 \left(\zeta_2 - \bar{\zeta}_2\right) \quad (8.126)$$

Using (8.125) and (8.126), we transform (8.123) into (8.109) and (8.110). Then, we obtain the normal forms (8.112) and (8.113) in the absence of autoparametric resonance and (8.116) and (8.117) in the presence of autoparametric resonance.

8.4 Exercises

8.4.1 Use the methods of multiple scales and normal forms to determine a first-order uniform expansion for

$$\ddot{u}_1 + \omega_1^2 u_1 = \alpha_1 u_1 u_2$$
$$\ddot{u}_2 + \omega_2^2 u_2 = \alpha_2 u_1^2$$

for small but finite amplitudes when $\omega_2 \approx 2\omega_1$.

8.4.2 Use the methods of multiple scales and normal forms to determine first-order uniform expansions for

$$\ddot{u}_1 + \omega_1^2 u_1 = \epsilon \alpha_1 u_1 u_2 + \epsilon k_1 \cos \Omega_1 t$$
$$\ddot{u}_2 + \omega_2^2 u_2 = \epsilon \alpha_2 u_1^2 + \epsilon k_2 \cos \Omega_2 t$$

when

a) $\omega_2 \approx 2\omega_1$ and $\Omega_1 \approx \omega_1$,
b) $\omega_2 \approx 2\omega_1$ and $\Omega_2 \approx \omega_2$.

8.4.3 Determine the normal form of

$$\ddot{u}_1 + \tfrac{1}{2}\dot{u}_2 + \delta u = \epsilon u_1 u_2$$
$$\ddot{u}_2 - \tfrac{1}{2}\dot{u}_1 + \tfrac{1}{2}u_2 = \epsilon u_1^2$$

when $\delta \approx 1/2$.

8.4.4 Determine the normal form of

$$\ddot{u}_1 + \omega_1^2 u_1 = \alpha_1 u_2 u_3$$
$$\ddot{u}_2 + \omega_2^2 u_2 = \alpha_2 u_1 u_3$$
$$\ddot{u}_3 + \omega_3^2 u_3 = \alpha_3 u_1 u_2$$

when $\omega_3 \approx \omega_2 + \omega_1$.

8.4.5 Determine the normal form of

$$\ddot{u}_1 + 2\dot{u}_2 + 3u_1 + 2\epsilon \cos \Omega t\, (f_{11}u_1 + f_{12}u_2) = 0$$
$$\ddot{u}_2 + \dot{u}_1 + 12u_2 + 2\epsilon \cos \Omega t\, (f_{21}u_1 + f_{22}u_2) = 0$$

when $\Omega \approx 5$ and $\Omega \approx 1$.

8.4.6 Determine the normal form of

$$\ddot{u}_1 + 4\dot{u}_2 + 3u_1 + 2 \cos \Omega t\, (f_{11}u_1 + f_{12}u_2) = 0$$
$$\ddot{u}_2 - \dot{u}_1 + 3u_2 + 2 \cos \Omega t\, (f_{21}u_1 + f_{22}u_2) = 0$$

when $\Omega \approx 4$ and $\Omega \approx 2$.

8.4.7 Determine the normal form of

$$\ddot{u}_1 + 5\dot{u}_2 + 6u_1 + 2 \cos \Omega t\, (f_{11}u_1 + f_{12}u_2) = 0$$
$$\ddot{u}_2 + \dot{u}_1 + 24u_2 + 2 \cos \Omega t\, (f_{21}u_1 + f_{22}u_2) = 0$$

when $\Omega \approx 7$ and $\Omega \approx 1$.

8.4.8 Use the methods of multiple scales and normal forms to determine an approximation to the solution of

$$\ddot{u}_1 + \omega_1^2 u_1 = \alpha_1 \dot{u}_1 \dot{u}_2$$
$$\ddot{u}_2 + \omega_2^2 u_2 = \alpha_2 \dot{u}_1^2$$
$$\omega_2 \approx 2\omega_1$$

8.4.9 Use the methods of multiple scales and normal forms to determine a first-order approximation to the solution of

$$\ddot{u}_1 + \omega_1^2 u_1 = \epsilon \left(\alpha_1 u_1^2 + \alpha_2 u_1 u_2 + \alpha_3 u_2^2\right)$$
$$\ddot{u}_2 + \omega_2^2 u_2 = \epsilon \left(\alpha_4 u_1^2 + \alpha_5 u_1 u_2 + \alpha_6 u_2^3\right)$$
$$\omega_2 \approx 2\omega_1$$

8.4.10 Use the methods of multiple scales and normal forms to determine a first-order approximation to

$$\ddot{u}_1 + \omega_1^2 u_1 + \epsilon \left(\delta_1 u_1^2 + 2\delta_2 u_1 u_2 + \delta_3 u_2^2\right) = F_1 \cos \Omega t$$
$$\ddot{u}_2 + \omega_2^2 u_2 + \epsilon \left(\delta_2 u_1^2 + 2\delta_3 u_1 u_2 + \delta_4 u_2^2\right) = 0$$

where

$$\omega_2 \approx 2\omega_1 \quad \text{and} \quad \Omega \approx 2\omega_1$$

8.4.11 Use the methods of multiple scales and normal forms to determine a first-order approximation to

$$\ddot{u}_1 + \omega_1^2 u_1 + \epsilon \left(\delta_1 u_1^2 + 2\delta_2 u_1 u_2 + \delta_3 u_2^2\right) = 2F_1 \cos \Omega t$$
$$\ddot{u}_2 + \omega_2^2 u_2 + \epsilon \left(\delta_2 u_1^2 + 2\delta_3 u_1 u_2 + \delta_4 u_2^2\right) = 0$$

where

$$\omega_2 \approx 2\omega_1 \quad \text{and} \quad \Omega \approx \omega_1$$

9
TDOF Systems with Cubic Nonlinearities

In this chapter, we consider two-degree-of-freedom nongyroscopic and gyroscopic systems with cubic nonlinearities. Nongyroscopic systems are treated in the following section, and gyroscopic systems are treated in Section 9.2.

9.1
Nongyroscopic Systems

We assume that the system can be modeled by (8.1), where A and D are 2×2 matrices and $\boldsymbol{x}$ and $\boldsymbol{N}$ are column vectors of length 2. Moreover, we assume that a linear transformation $\boldsymbol{x} = P\boldsymbol{u}$ has been introduced so that (8.1) becomes (8.2) and that J and D are diagonal. The case in which J has a generic nonsemisimple structure is treated in Sections 9.1.5 and 9.1.6. Thus, we rewrite (8.2) as

$$\ddot{u}_m + \omega_m^2 u_m + 2\epsilon\mu_m \dot{u}_m = \epsilon \frac{\partial V}{\partial u_m}(u_1, u_2) + 2 f_m \cos(\Omega t + \tau_m) \tag{9.1}$$

where

$$V = \delta_1 u_1^4 + \delta_2 u_1^3 u_2 + \delta_3 u_1^2 u_2^2 + \delta_4 u_1 u_2^3 + \delta_5 u_2^4 \tag{9.2}$$

Again, as a first step, we cast (9.1) in complex-valued form using the transformation

$$u_m = \zeta_m + \bar{\zeta}_m , \quad \dot{u}_m = i\omega_m(\zeta_m - \bar{\zeta}_m)$$
$$\dot{z}_m = i\Omega z_m , \quad z_m = f_m e^{i(\Omega t + \tau_m)}$$

and obtain

$$\dot{\zeta}_m = i\omega_m \zeta_m - \epsilon\mu_m(\zeta_m - \bar{\zeta}_m) - \frac{i}{2\omega_m}(z_m + \bar{z}_m) - \frac{i\epsilon}{2\omega_m}\frac{\partial V}{\partial u_m}(\zeta_1 + \bar{\zeta}_1, \zeta_2 + \bar{\zeta}_2) \tag{9.3}$$

Before proceeding further, we must distinguish between two cases: $\Omega \approx \omega_m$ (corresponding to the primary resonance of the mth mode) and Ω is away from both

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ω_1 and ω_2. We start with the latter case and consider the primary-resonance case in Section 9.1.4.

When Ω is away from ω_1 and ω_2, we use the transformation (8.8) and (8.10) and rewrite (9.3) as

$$
\begin{aligned}
\dot{\eta}_1 = {} & i\omega_1\eta_1 - \epsilon\mu_1(\eta_1 - \bar{\eta}_1) - \frac{i\epsilon}{2\omega_1}\left[4\delta_1\left(\eta_1 + \bar{\eta}_1 + \frac{z_1 + \bar{z}_1}{\omega_1^2 - \Omega^2}\right)^3\right. \\
& + 3\delta_2\left(\eta_1 + \bar{\eta}_1 + \frac{z_1 + \bar{z}_1}{\omega_1^2 - \Omega^2}\right)^2\left(\eta_2 + \bar{\eta}_2 + \frac{z_2 + \bar{z}_2}{\omega_2^2 - \Omega^2}\right) \\
& + 2\delta_3\left(\eta_1 + \bar{\eta}_1 + \frac{z_1 + \bar{z}_2}{\omega_1^2 - \Omega^2}\right)\left(\eta_2 + \bar{\eta}_2 + \frac{z_2 + \bar{z}_2}{\omega_2^2 - \Omega^2}\right)^2 \\
& \left. + \delta_4\left(\eta_2 + \bar{\eta}_2 + \frac{z_2 + \bar{z}_2}{\omega_2^2 - \Omega^2}\right)^3\right]
\end{aligned}
\tag{9.4}
$$

$$
\begin{aligned}
\dot{\eta}_2 = {} & i\omega_2\eta_2 - \epsilon\mu_2(\eta_2 - \bar{\eta}_2) - \frac{i\epsilon}{2\omega_2}\left[\delta_2\left(\eta_1 + \bar{\eta}_1 + \frac{z_1 + \bar{z}_1}{\omega_1^2 - \Omega^2}\right)^3\right. \\
& + 2\delta_3\left(\eta_1 + \bar{\eta}_1 + \frac{z_1 + \bar{z}_1}{\omega_1^2 - \Omega^2}\right)^2\left(\eta_2 + \bar{\eta}_2 + \frac{z_2 + \bar{z}_2}{\omega_2^2 - \Omega^2}\right) \\
& + 3\delta_4\left(\eta_1 + \bar{\eta}_1 + \frac{z_1 + \bar{z}_2}{\omega_1^2 - \Omega^2}\right)\left(\eta_2 + \bar{\eta}_2 + \frac{z_2 + \bar{z}_2}{\omega_2^2 - \Omega^2}\right)^2 \\
& \left. + 4\delta_5\left(\eta_2 + \bar{\eta}_2 + \frac{z_2 + \bar{z}_2}{\omega_2^2 - \Omega^2}\right)^3\right]
\end{aligned}
\tag{9.5}
$$

In addition to the resonance terms η_1, $\eta_1^2\bar{\eta}_1$, and $\eta_2\bar{\eta}_2\eta_1$ in (9.4) and the resonance terms η_2, $\eta_2^2\bar{\eta}_2$, and $\eta_1\bar{\eta}_1\eta_2$ in (9.5), near-resonance terms occur when

- $\omega_2 \approx 3\omega_1$: Three-to-one internal resonance,
- $\omega_2 \approx \omega_1$: One-to-one internal resonance,
- $\Omega \approx 3\omega_m$: Subharmonic resonance of order one-third,
- $\Omega \approx 1/3\omega_m$: Superharmonic resonance of order three,
- $\Omega \approx \omega_m \pm \omega_n$: Combination resonance.

Next, we present the normal forms of (9.4) and (9.5) for several combinations of the internal and external resonances.

9.1.1
The Case of No Internal Resonances

Substituting the near-identity transformation (8.14) into (9.4) and (9.5) and choosing the h_m to eliminate the nonresonance terms, we obtain the following normal forms:

No External Resonance

$$\dot{\xi}_1 = i\omega_1\xi_1 - \frac{i\epsilon\sigma_1}{2\omega_1}\xi_1 - \epsilon\mu_1\xi_1 - \frac{i\epsilon}{2\omega_1}\left[12\delta_1\xi_1^2\bar{\xi}_1 + 4\delta_3\xi_2\bar{\xi}_2\xi_1\right] \tag{9.6}$$

$$\dot{\xi}_2 = i\omega_2\xi_2 - \frac{i\epsilon\sigma_2}{2\omega_2}\xi_2 - \epsilon\mu_2\xi_2 - \frac{i\epsilon}{2\omega_2}\left[4\delta_3\xi_1\bar{\xi}_1\xi_2 + 12\delta_5\xi_2^2\bar{\xi}_2\right] \tag{9.7}$$

where

$$\sigma_1 = \frac{24\delta_1 z_1\bar{z}_1}{(\omega_1^2 - \Omega^2)^2} + \frac{6\delta_2(z_1\bar{z}_2 + z_2\bar{z}_1)}{(\omega_1^2 - \Omega^2)(\omega_2^2 - \Omega^2)} + \frac{4\delta_3 z_2\bar{z}_2}{(\omega_2^2 - \Omega^2)^2} \tag{9.8}$$

$$\sigma_2 = \frac{24\delta_5 z_2\bar{z}_2}{(\omega_2^2 - \Omega^2)^2} + \frac{6\delta_4(z_1\bar{z}_2 + z_2\bar{z}_1)}{(\omega_1^2 - \Omega^2)(\omega_2^2 - \Omega^2)} + \frac{4\delta_3 z_1\bar{z}_1}{(\omega_1^2 - \Omega^2)^2} \tag{9.9}$$

$\Omega \approx 3\omega_2$

$$\dot{\xi}_1 = i\omega_1\xi_1 - \frac{i\epsilon\sigma_1}{2\omega_1}\xi_1 - \epsilon\mu_1\xi_1 - \frac{i\epsilon}{2\omega_1}\left[12\delta_1\xi_1^2\bar{\xi}_1 + 4\delta_3\xi_2\bar{\xi}_2\xi_1\right] \tag{9.10}$$

$$\dot{\xi}_2 = i\omega_2\xi_2 - \frac{i\epsilon\sigma_2}{2\omega_2}\xi_2 - \epsilon\mu_2\xi_2 - \frac{i\epsilon}{2\omega_2}\left[4\delta_3\xi_1\bar{\xi}_1\xi_2 + 12\delta_5\xi_2^2\bar{\xi}_2\right] - i\epsilon\Gamma_1\bar{\xi}_2^2 \tag{9.11}$$

where σ_1 and σ_2 are defined in (9.8) and (9.9) and

$$\Gamma_1 = \frac{1}{2\omega_2}\left[\frac{3\delta_4 z_1}{\omega_1^2 - \Omega^2} + \frac{12\delta_5 z_2}{\omega_2^2 - \Omega^2}\right] \tag{9.12}$$

$\Omega \approx 1/3\omega_1$

$$\dot{\xi}_1 = i\omega_1\xi_1 - \frac{i\epsilon\sigma_1}{2\omega_1}\xi_1 - \epsilon\mu_1\xi_1 - \frac{i\epsilon}{2\omega_1}\left[12\delta_1\xi_1^2\bar{\xi}_1 + 4\delta_3\xi_2\bar{\xi}_2\xi_1\right] - i\epsilon\Gamma_2 \tag{9.13}$$

$$\dot{\xi}_2 = i\omega_2\xi_2 - \frac{i\epsilon\sigma_2}{2\omega_2}\xi_2 - \epsilon\mu_2\xi_2 - \frac{i\epsilon}{2\omega_2}\left[4\delta_3\xi_1\bar{\xi}_1\xi_2 + 12\delta_5\xi_2^2\bar{\xi}_2\right] \tag{9.14}$$

where σ_1 and σ_2 are defined in (9.8) and (9.9) and

$$\Gamma_2 = \frac{1}{2\omega_1}\left[\frac{4\delta_1 z_1^3}{(\omega_1^2 - \Omega^2)^3} + \frac{3\delta_2 z_1^2 z_2}{(\omega_1^2 - \Omega^2)^2(\omega_2^2 - \Omega^2)} + \frac{2\delta_3 z_1 z_2^2}{(\omega_1^2 - \Omega^2)(\omega_2^2 - \Omega^2)^2} + \frac{\delta_4 z_2^3}{(\omega_2^2 - \Omega^2)^3}\right] \tag{9.15}$$

$\Omega \approx \omega_2 + 2\omega_1$

$$\dot{\xi}_1 = i\omega_1\xi_1 - \frac{i\epsilon\sigma_1}{2\omega_1}\xi_1 - \epsilon\mu_1\xi_1 - \frac{i\epsilon}{2\omega_1}\left[12\delta_1\xi_1^2\bar{\xi}_1 + 4\delta_3\xi_2\bar{\xi}_2\xi_1\right] - i\epsilon\Gamma_3\bar{\xi}_2\bar{\xi}_1 \tag{9.16}$$

$$\dot{\xi}_2 = i\omega_2\xi_2 - \frac{i\epsilon\sigma_2}{2\omega_2}\xi_2 - \epsilon\mu_2\xi_2 - \frac{i\epsilon}{2\omega_2}\left[4\delta_3\xi_1\bar{\xi}_1\xi_2 + 12\delta_5\xi_2^2\bar{\xi}_2\right] - i\epsilon\Gamma_4\bar{\xi}_1^2 \tag{9.17}$$

where σ_1 and σ_2 are defined in (9.8) and (9.9) and

$$\Gamma_3 = \frac{1}{\omega_1}\left[\frac{3\delta_2 z_1}{\omega_1^2 - \Omega^2} + \frac{2\delta_3 z_2}{\omega_2^2 - \Omega^2}\right] \tag{9.18}$$

$$\Gamma_4 = \frac{1}{2\omega_2}\left[\frac{3\delta_2 z_1}{\omega_1^2 - \Omega^2} + \frac{2\delta_3 z_2}{\omega_2^2 - \Omega^2}\right] \tag{9.19}$$

9.1.2
Three-to-One Autoparametric Resonance

The three-to-one internal resonance $\omega_2 \approx 3\omega_1$ produces the near-resonance term $\eta_2\bar{\eta}_1^2$ in (9.4) and the near-resonance term η_1^3 in (9.5). Hence, substituting the near-identity transformation (8.14) into (9.4) and (9.5) and choosing the h_m to eliminate the nonresonance terms, we obtain the following normal forms:

No External Resonance

$$\dot{\xi}_1 = i\omega_1\xi_1 - \frac{i\epsilon\sigma_1}{2\omega_1}\xi_1 - \epsilon\mu_1\xi_1 - \frac{i\epsilon}{2\omega_1}\left[12\delta_1\xi_1^2\bar{\xi}_1 + 4\delta_3\xi_2\bar{\xi}_2\xi_1 + 3\delta_2\xi_2\bar{\xi}_1^2\right] \tag{9.20}$$

$$\dot{\xi}_2 = i\omega_2\xi_2 - \frac{i\epsilon\sigma_2}{2\omega_2}\xi_2 - \epsilon\mu_2\xi_2 - \frac{i\epsilon}{2\omega_2}\left[4\delta_3\xi_1\bar{\xi}_1\xi_2 + 12\delta_5\xi_2^2\bar{\xi}_2 + \delta_2\xi_1^3\right] \tag{9.21}$$

where σ_1 and σ_2 are defined in (9.8) and (9.9).

$\Omega \approx 2\omega_2 - \omega_1$

$$\begin{aligned}\dot{\xi}_1 = {} & i\omega_1\xi_1 - \frac{i\epsilon\sigma_1}{2\omega_1}\xi_1 - \epsilon\mu_1\xi_1 \\ & - \frac{i\epsilon}{2\omega_1}\left[12\delta_1\xi_1^2\bar{\xi}_1 + 4\delta_3\xi_2\bar{\xi}_2\xi_1 + 3\delta_2\xi_2\bar{\xi}_1^2\right] - i\epsilon\Gamma_5\xi_2^2\end{aligned} \tag{9.22}$$

$$\begin{aligned}\dot{\xi}_2 = {} & i\omega_2\xi_2 - \frac{i\epsilon\sigma_2}{2\omega_2}\xi_2 - \epsilon\mu_2\xi_2 \\ & - \frac{i\epsilon}{2\omega_2}\left[4\delta_3\xi_1\bar{\xi}_1\xi_2 + 12\delta_5\xi_2^2\bar{\xi}_2 + \delta_2\xi_1^3\right] - i\epsilon\Gamma_6\xi_1\bar{\xi}_2\end{aligned} \tag{9.23}$$

where

$$\Gamma_5 = \frac{1}{2\omega_1}\left[\frac{2\delta_3\bar{z}_1}{\omega_1^2 - \Omega^2} + \frac{3\delta_4\bar{z}_2}{\omega_2^2 - \Omega^2}\right] \tag{9.24}$$

$$\Gamma_6 = \frac{1}{\omega_2}\left[\frac{2\delta_3 z_1}{\omega_1^2 - \Omega^2} + \frac{3\delta_4 z_2}{\omega_2^2 - \Omega^2}\right] \tag{9.25}$$

The normal forms corresponding to the other external resonances can be obtained by simply adding the appropriate Γ_m terms to the right-hand sides of (9.20) and (9.21), as in Section 9.1.1.

9.1.3
One-to-One Internal Resonance

The one-to-one internal resonance corresponding to $\omega_2 \approx \omega_1$ produces the near-resonance terms $\eta_2^2\bar{\eta}_2, \eta_1^2\bar{\eta}_2, \eta_1\bar{\eta}_1\eta_2$, and $\eta_2^2\bar{\eta}_1$ in (9.4) and the near-resonance terms $\eta_1^2\bar{\eta}_1, \eta_2\bar{\eta}_2\eta_1, \eta_2^2\bar{\eta}_1$, and $\eta_1^2\bar{\eta}_2$ in (9.5). Consequently, substituting the near-identity transformation (8.14) into (9.4) and (9.5) and choosing the h_m to eliminate the nonresonance terms, we obtain the following normal form in the absence of external resonances:

$$\dot{\xi}_1 = i\omega_1\xi_1 - \frac{i\epsilon\sigma_1}{2\omega_1}\xi_1 - \frac{i\epsilon\sigma_3}{2\omega_1}\xi_2 - \epsilon\mu_1\xi_1 - \frac{i\epsilon}{2\omega_1}\left[3\delta_2\left(2\xi_1\bar{\xi}_1\xi_2 + \xi_1^2\bar{\xi}_2\right)\right.$$
$$\left. + 12\delta_1\xi_1^2\bar{\xi}_1 + 2\delta_3\left(2\xi_2\bar{\xi}_2\xi_1 + \xi_2^2\bar{\xi}_1\right) + 3\delta_4\xi_2^2\bar{\xi}_2\right] \quad (9.26)$$

$$\dot{\xi}_2 = i\omega_2\xi_2 - \frac{i\epsilon\sigma_2}{2\omega_2}\xi_2 - \frac{i\epsilon\sigma_3}{2\omega_2}\xi_1 - \epsilon\mu_2\xi_2 - \frac{i\epsilon}{2\omega_2}\left[2\delta_3\left(2\xi_1\bar{\xi}_1\xi_2 + \xi_1^2\bar{\xi}_2\right)\right.$$
$$\left. + 3\delta_2\xi_1^2\bar{\xi}_1 + 3\delta_4\left(2\xi_2\bar{\xi}_2\xi_1 + \xi_2^2\bar{\xi}_1\right) + 12\delta_5\xi_2^2\bar{\xi}_2\right] \quad (9.27)$$

where

$$\sigma_3 = \frac{6\delta_2 z_1\bar{z}_1}{(\omega_1^2 - \Omega^2)^2} + \frac{4\delta_3(z_1\bar{z}_2 + \bar{z}_1 z_2)}{(\omega_1^2 - \Omega^2)(\omega_2^2 - \Omega^2)} + \frac{6\delta_4 z_2\bar{z}_2}{(\omega_2^2 - \Omega^2)^2} \quad (9.28)$$

The normal forms of (9.4) and (9.5) in the presence of external resonances can be obtained by simply adding the appropriate Γ_m terms to the right-hand sides of (9.26) and (9.27), as in Sections 9.1.1 and 9.1.2. However, because the analysis in this section is valid only when Ω is away from ω_1 and ω_2, one needs to make sure that the external resonance does not produce a primary resonance.

9.1.4
Primary Resonances

Primary resonances can easily be treated by scaling the excitation at $O(\epsilon)$ so that its effect balances the effects of the damping and the nonlinearity. Consequently, substituting (8.60) into (9.3), using (9.2), and choosing the h_m to eliminate the nonresonance terms, we obtain the following normal forms:

No Internal Resonances, $\Omega \approx \omega_n$

$$\dot{\xi}_1 = i\omega_1\xi_1 - \epsilon\mu_1\xi_1 - \frac{i\epsilon}{2\omega_1}\left[12\delta_1\xi_1^2\bar{\xi}_1 + 4\delta_3\xi_2\bar{\xi}_2\xi_1\right] - \frac{i\epsilon}{2\omega_1}z_1\delta_{1n} \quad (9.29)$$

$$\dot{\xi}_2 = i\omega_2\xi_2 - \epsilon\mu_2\xi_2 - \frac{i\epsilon}{2\omega_2}\left[4\delta_3\xi_1\bar{\xi}_1\xi_2 + 12\delta_5\xi_2^2\bar{\xi}_2\right] - \frac{i\epsilon}{2\omega_2}z_2\delta_{2n} \quad (9.30)$$

Three-to-One Internal Resonance, $\Omega \approx \omega_n$

$$\dot{\xi}_1 = i\omega_1\xi_1 - \epsilon\mu_1\xi_1 - \frac{i\epsilon}{2\omega_1}\left[12\delta_1\xi_1^2\bar{\xi}_1 + 4\delta_3\xi_2\bar{\xi}_2\xi_1 + 3\delta_2\xi_2\bar{\xi}_1^2\right] - \frac{i\epsilon}{2\omega_1}z_1\delta_{1n} \tag{9.31}$$

$$\dot{\xi}_2 = i\omega_2\xi_2 - \epsilon\mu_2\xi_2 - \frac{i\epsilon}{2\omega_2}\left[4\delta_3\xi_1\bar{\xi}_1\xi_2 + 12\delta_5\xi_2^2\bar{\xi}_2 + \delta_2\xi_1^3\right] - \frac{i\epsilon}{2\omega_2}z_2\delta_{2n} \tag{9.32}$$

One-to-One Internal Resonance, $\Omega \approx \omega_n$

$$\begin{aligned}\dot{\xi}_1 = {} & i\omega_1\xi_1 - \epsilon\mu_1\xi_1 - \frac{i\epsilon}{2\omega_1}\left[12\delta_1\xi_1^2\bar{\xi}_1 + 3\delta_2\left(2\xi_1\bar{\xi}_1\xi_2 + \xi_1^2\bar{\xi}_2\right)\right.\\ & \left.+2\delta_3\left(2\xi_2\bar{\xi}_2\xi_1 + \xi_2^2\bar{\xi}_1\right) + 3\delta_4\xi_2^2\bar{\xi}_2\right] - \frac{i\epsilon}{2\omega_1}z_1\delta_{1n}\end{aligned} \tag{9.33}$$

$$\begin{aligned}\dot{\xi}_2 = {} & i\omega_2\xi_2 - \epsilon\mu_2\xi_2 - \frac{i\epsilon}{2\omega_2}\left[3\delta_2\xi_1^2\bar{\xi}_1 + 2\delta_3\left(2\xi_1\bar{\xi}_1\xi_2 + \xi_1^2\bar{\xi}_2\right)\right.\\ & \left.+3\delta_4\left(2\xi_2\bar{\xi}_2\xi_1 + \xi_2^2\bar{\xi}_1\right) + 12\delta_5\xi_2^2\bar{\xi}_2\right] - \frac{i\epsilon}{2\omega_2}z_2\delta_{2n}\end{aligned} \tag{9.34}$$

9.1.5
A Nonsemisimple One-to-One Internal Resonance

In this section, we treat a two-degree-of-freedom system whose linear undamped operator has a generic nonsemisimple structure; that is, we treat the system

$$\ddot{u}_1 + \omega^2 u_1 = -2\mu_1\dot{\mu}_1 - \alpha_{11}u_1^3 - \alpha_{12}u_1^2u_2 - \alpha_{13}u_1u_2^2 - \alpha_{14}u_2^3 \tag{9.35}$$

$$\ddot{u}_2 + \omega^2 u_2 = -2\mu_2\dot{\mu}_2 - u_1 - \alpha_{21}u_1^3 - \alpha_{22}u_1^2u_2 - \alpha_{23}u_1u_2^2 - \alpha_{24}u_2^3 \tag{9.36}$$

Again, as a first step in the application of the method of normal forms, we use the transformation

$$u_1 = \zeta_1 + \bar{\zeta}_1\,, \quad \dot{u}_1 = i\omega\left(\zeta_1 - \bar{\zeta}_1\right) \tag{9.37}$$

$$u_2 = \zeta_2 + \bar{\zeta}_2\,, \quad \dot{u}_2 = i\omega\left(\zeta_2 - \bar{\zeta}_2\right) \tag{9.38}$$

to recast (9.35) and (9.36) in the complex-valued form

$$\begin{aligned}\dot{\zeta}_1 = {} & i\omega\zeta_1 - \mu_1\left(\zeta_1 - \bar{\zeta}_1\right)\\ & + \frac{i}{2\omega}\left[\alpha_{11}\left(\zeta_1 + \bar{\zeta}_1\right)^3 + \alpha_{12}\left(\zeta_1 + \bar{\zeta}_1\right)^2\left(\zeta_2 + \bar{\zeta}_2\right)\right.\\ & \left.+\alpha_{13}\left(\zeta_1 + \bar{\zeta}_1\right)\left(\zeta_2 + \bar{\zeta}_2\right)^2 + \alpha_{14}\left(\zeta_2 + \bar{\zeta}_2\right)^3\right]\end{aligned} \tag{9.39}$$

$$\dot{\zeta}_2 = i\omega\zeta_2 - \mu_2\left(\zeta_2 - \bar{\zeta}_2\right) + \frac{i}{2\omega}\left(\zeta_1 + \bar{\zeta}_1\right) + \frac{i}{2\omega}\Big[\alpha_{21}\left(\zeta_1 + \bar{\zeta}_1\right)^3 + \alpha_{22}\left(\zeta_1 + \bar{\zeta}_1\right)^2\left(\zeta_2 + \bar{\zeta}_2\right) + \alpha_{23}\left(\zeta_1 + \bar{\zeta}_1\right)\left(\zeta_2 + \bar{\zeta}_2\right)^2 + \alpha_{24}\left(\zeta_2 + \bar{\zeta}_2\right)^3\Big] \tag{9.40}$$

In determining the normal form of (9.39) and (9.40), one can follow one of two approaches. In the first approach, which we followed in Sections 7.3.1–7.3.3, we determine the normal form assuming that ζ_1 and ζ_2 are the same order and then we use the fact that ζ_2 is much larger than ζ_1 to simplify the resulting forms. In the second approach, we use the fact that ζ_2 is much larger than ζ_1 to first simplify (9.39) and (9.40) and then determine the normal form of the simplified equations.

Following the first approach, we note that ζ_1, ζ_2, $\zeta_1^2\bar{\zeta}_1$, $\zeta_1^2\bar{\zeta}_2$, $\zeta_1\bar{\zeta}_1\zeta_2$, $\zeta_2^2\bar{\zeta}_2$, $\zeta_2\bar{\zeta}_2\zeta_1$, and $\zeta_2^2\bar{\zeta}_2$ are resonance terms. Hence, introducing the near-identity transformation

$$\zeta_m = \eta_m + h_m\left(\eta_1, \eta_2, \bar{\eta}_1, \bar{\eta}_2\right) \tag{9.41}$$

into (9.39) and (9.40) and choosing h_1 and h_2 to eliminate the nonresonance terms, we obtain the normal form

$$\dot{\eta}_1 = i\omega\eta_1 - \mu_1\eta_1 + \frac{i}{2\omega}\left[3\alpha_{11}\eta_1^2\bar{\eta}_1 + 2\alpha_{12}\eta_1\bar{\eta}_1\eta_2 + \alpha_{12}\eta_1^2\bar{\eta}_2 + \alpha_{13}\eta_2^2\bar{\eta}_1 + 2\alpha_{13}\eta_1\eta_2\bar{\eta}_2 + 3\alpha_{14}\eta_2^2\bar{\eta}_2\right] \tag{9.42}$$

$$\dot{\eta}_2 = i\omega\eta_2 - \mu_2\eta_2 + \frac{i}{2\omega}\eta_1 + \frac{i}{2\omega}\left[3\alpha_{21}\eta_1^2\bar{\eta}_1 + 2\alpha_{22}\eta_1\bar{\eta}_1\eta_2 + \alpha_{22}\eta_1^2\bar{\eta}_2 + 2\alpha_{23}\eta_2\bar{\eta}_2\eta_1 + \alpha_{23}\eta_2^2\bar{\eta}_1 + 3\alpha_{24}\eta_2^2\bar{\eta}_2\right] \tag{9.43}$$

Next, we use the fact that η_2 is much larger than η_1 and introduce new variables defined by

$$\eta_1 = \epsilon\xi_1 \quad \text{and} \quad \eta_2 = \epsilon^{1-\lambda_2}\xi_2 \tag{9.44}$$

where ξ_1 and ξ_2 are $O(1)$ and $\lambda_2 > 0$. Moreover, we assume that the damping is weak and hence scale the μ_n as

$$\mu_n = \epsilon^{\lambda_1}\hat{\mu}_n \tag{9.45}$$

where the $\hat{\mu}_n$ are $O(1)$. Substituting (9.44) and (9.45) into (9.42) and (9.43), we have

$$\dot{\xi}_1 = i\omega\xi_1 - \epsilon^{\lambda_1}\hat{\mu}_1\xi_1 + \frac{i}{2\omega}\Big[3\epsilon^2\alpha_{11}\xi_1^2\bar{\xi}_1 + 2\epsilon^{2-\lambda_2}\alpha_{12}\xi_1\bar{\xi}_1\xi_2 + \epsilon^{2-\lambda_2}\alpha_{12}\xi_1^2\bar{\xi}_2 + \epsilon^{2-2\lambda_2}\alpha_{13}\xi_2^2\bar{\xi}_1 + 2\epsilon^{2-2\lambda_2}\alpha_{13}\xi_1\xi_2\bar{\xi}_2 + 3\epsilon^{2-3\lambda_2}\alpha_{14}\xi_2^2\bar{\xi}_2\Big] \tag{9.46}$$

$$\dot{\xi}_2 = i\omega\xi_2 - \epsilon^{\lambda_1}\hat{\mu}_2\xi_2 + \frac{i}{2\omega}\epsilon^{\lambda_2}\xi_1 + \frac{i}{2\omega}\Big[3\epsilon^{2+\lambda_2}\alpha_{21}\xi_1^2\bar{\xi}_1 + 2\epsilon^2\alpha_{22}\xi_1\bar{\xi}_1\xi_2 + \epsilon^2\alpha_{22}\xi_1^2\bar{\xi}_2 + 2\epsilon^{2-\lambda_2}\alpha_{23}\xi_2\bar{\xi}_2\xi_1 + \epsilon^{2-\lambda_2}\alpha_{23}\xi_2^2\bar{\xi}_1 + 3\epsilon^{2-2\lambda_2}\alpha_{24}\xi_2^2\bar{\xi}_2\Big] \tag{9.47}$$

Balancing the damping term and the dominant nonlinear term in (9.46), namely, the term proportional to α_{14}, we have

$$\lambda_1 = 2 - 3\lambda_2 \tag{9.48}$$

Balancing the damping term and the term that causes the instability in (9.47), namely, the term $i(2\omega)^{-1}\epsilon^{\lambda_2}\xi_1$, we have

$$\lambda_1 = \lambda_2 \tag{9.49}$$

We note that due to (9.48), the dominant nonlinear term in (9.47), namely, the term proportional to α_{24}, is small compared with the damping term. Solving (9.48) and (9.49), we obtain

$$\lambda_1 = \lambda_2 = \tfrac{1}{2} \tag{9.50}$$

and hence (9.46) and (9.47) simplify to

$$\dot{\xi}_1 = i\omega\xi_1 - \epsilon^{1/2}\hat{\mu}_1\xi_1 + \frac{3i\epsilon^{1/2}}{2\omega}\alpha_{14}\xi_2^2\bar{\xi}_2 + \cdots \tag{9.51}$$

$$\dot{\xi}_2 = i\omega\xi_2 - \epsilon^{1/2}\hat{\mu}_2\xi_2 + \frac{i\epsilon^{1/2}}{2\omega}\xi_1 + \cdots \tag{9.52}$$

Equations 9.51 and 9.52 agree with those obtained by Tezak, Nayfeh, and Mook (1982) by using the method of multiple scales.

In the second approach, we use the fact that the damping and nonlinearity are weak and that ζ_2 is much larger than ζ_1 to first simplify (9.39) and (9.40) and then determine the normal form of the simplified equations. If we assume that $\zeta_1 = O(\epsilon)$, where ϵ is a small nondimensional parameter, then $\zeta_2 = O(\epsilon^{1-\lambda_2})$, where $\lambda_2 > 0$. Moreover, for the case of weak damping, $\mu_m = O(\epsilon^{\lambda_1})$, where λ_1 is positive. Using these scalings, we introduce new scaled variables defined by $\mu_m = \epsilon^{\lambda_1}\hat{\mu}_n$, $\zeta_1 = \epsilon\eta_1$, and $\zeta_2 = \epsilon^{1-\lambda_2}\eta_2$, where the η_n and $\hat{\mu}_n$ are $O(1)$, in (9.39) and (9.40) and obtain

$$\dot{\eta}_1 = i\omega\eta_1 - \epsilon^{\lambda_1}\hat{\mu}_1(\eta_1 - \bar{\eta}_1) + \frac{i}{2\omega}\Big[\epsilon^2\alpha_{11}(\eta_1 + \bar{\eta}_1)^3 + \epsilon^{2-3\lambda_2}\alpha_{14}(\eta_2 + \bar{\eta}_2)^3\Big] + \epsilon^{2-\lambda_2}\alpha_{12}(\eta_1 + \bar{\eta}_1)^2(\eta_2 + \bar{\eta}_2) + \epsilon^{2-2\lambda_2}\alpha_{13}(\eta_1 + \bar{\eta}_1)(\eta_2 + \bar{\eta}_2)^2 \tag{9.53}$$

$$\dot{\eta}_2 = i\omega\eta_2 - \epsilon^{\lambda_1}\hat{\mu}_2(\eta_2 - \bar{\eta}_2) + \frac{i}{2\omega}\epsilon^{\lambda_2}(\eta_1 + \bar{\eta}_1)$$
$$+ \frac{i}{2\omega}\Big[\epsilon^{2+\lambda_2}\alpha_{21}(\eta_1+\bar{\eta}_1)^3 + \epsilon^2\alpha_{22}(\eta_1+\bar{\eta}_1)^2(\eta_2+\bar{\eta}_2)$$
$$+\epsilon^{2-\lambda_2}\alpha_{23}(\eta_1+\bar{\eta}_1)(\eta_2+\bar{\eta}_2)^2 + \epsilon^{2-2\lambda_2}\alpha_{24}(\eta_2+\bar{\eta}_2)^3\Big] \tag{9.54}$$

Balancing the damping term and the dominant nonlinear term in (9.53), namely, the term proportional to α_{14}, we have

$$\lambda_1 = 2 - 3\lambda_2 \tag{9.55}$$

Balancing the damping term and the source of the instability in (9.54), namely, the term $i(2\omega)^{-1}\epsilon^{\lambda_2}(\eta_1 + \bar{\eta}_1)$, we have

$$\lambda_1 = \lambda_2 \tag{9.56}$$

Solving (9.55) and (9.56) yields

$$\lambda_1 = \lambda_2 = \tfrac{1}{2} \tag{9.57}$$

Hence, (9.53) and (9.54) become

$$\dot{\eta}_1 = i\omega\eta_1 - \epsilon^{1/2}\hat{\mu}_1(\eta_1 - \bar{\eta}_1) + \frac{i\epsilon^{1/2}}{2\omega}\alpha_{14}(\eta_2+\bar{\eta}_2)^3 + \cdots \tag{9.58}$$

$$\dot{\eta}_2 = i\omega\eta_2 - \epsilon^{1/2}\hat{\mu}_2(\eta_2 - \bar{\eta}_2) + \frac{i\epsilon^{1/2}}{2\omega}(\eta_1+\bar{\eta}_1) + \cdots \tag{9.59}$$

To simplify (9.58) and (9.59), we introduce the near-identity transformation

$$\eta_m = \xi_m + \epsilon^{1/2}h_m\left(\xi_1, \xi_2, \bar{\xi}_1, \bar{\xi}_2\right) + \cdots \tag{9.60}$$

and choose h_1 and h_2 to eliminate the resonance terms. We note that the terms proportional to η_1 and $\eta_2^2\bar{\eta}_2$ are resonance terms in (9.58), and hence, we can choose h_1 to eliminate all of the other terms, resulting in the normal form

$$\dot{\xi}_1 = i\omega\xi_1 - \epsilon^{1/2}\hat{\mu}_1\xi_1 + \frac{3i\epsilon^{1/2}}{2\omega}\alpha_{14}\xi_2^2\bar{\xi}_2 + \cdots \tag{9.61}$$

Similarly, we note that η_2 and η_1 are resonance terms in (9.59), and hence, we can choose h_2 to eliminate all of the other terms, resulting in the normal form

$$\dot{\xi}_2 = i\omega\xi_2 - \epsilon^{1/2}\hat{\mu}_2\xi_2 + \frac{i\epsilon^{1/2}}{2\omega}\xi_1 + \cdots \tag{9.62}$$

Equations 9.61 and 9.62 agree with (9.51) and (9.52) and with those obtained by Tezak, Nayfeh, and Mook (1982) by using the method of multiple scales.

9.1.6
A Parametrically Excited System with a Nonsemisimple Linear Structure

We consider the parametrically excited two-degree-of-freedom system

$$\ddot{u}_1 + \omega^2 u_1 + 2\mu_1 \dot{u}_1 + \alpha_{11} u_1^3 + \alpha_{12} u_1^2 u_2 + \alpha_{13} u_1 u_2^2 + \alpha_{14} u_2^3 + 2\left(F_{11} u_1 + F_{12} u_2\right) \cos \Omega t = 0 \tag{9.63}$$

$$\ddot{u}_2 + \omega^2 u_2 + 2\mu_2 \dot{u}_2 + u_1 + \alpha_{21} u_1^3 + \alpha_{22} u_1^2 u_2 + \alpha_{23} u_1 u_2^2 + \alpha_{24} u_2^3 + 2\left(F_{21} u_1 + F_{22} u_2\right) \cos \Omega t = 0 \tag{9.64}$$

when the damping and nonlinearities are weak. Again, as a first step, we transform (9.63) and (9.64) into two first-order complex-valued equations. Using (9.37) and (9.38) and the transformation

$$z = e^{i\Omega t} \tag{9.65}$$

we rewrite (9.63) and (9.64) as

$$\begin{aligned}\dot{\zeta}_1 = {} & i\omega\zeta_1 - \mu_1\left(\zeta_1 - \bar{\zeta}_1\right) + \frac{i}{2\omega}\Big[\alpha_{11}\left(\zeta_1 + \bar{\zeta}_1\right)^3 + \alpha_{14}\left(\zeta_2 + \bar{\zeta}_2\right)^3 \\ & +\alpha_{12}\left(\zeta_1 + \bar{\zeta}_1\right)^2\left(\zeta_2 + \bar{\zeta}_2\right) + \alpha_{13}\left(\zeta_1 + \bar{\zeta}_1\right)\left(\zeta_2 + \bar{\zeta}_2\right)^2\Big] \\ & + \frac{i}{2\omega}\left[F_{11}\left(\zeta_1 + \bar{\zeta}_1\right) + F_{12}\left(\zeta_2 + \bar{\zeta}_2\right)\right](z + \bar{z})\end{aligned} \tag{9.66}$$

$$\begin{aligned}\dot{\zeta}_2 = {} & i\omega\zeta_2 - \mu_2\left(\zeta_2 - \bar{\zeta}_2\right) + \frac{i}{2\omega}\Big[\zeta_1 + \bar{\zeta}_1 + \alpha_{22}\left(\zeta_1 + \bar{\zeta}_1\right)^2\left(\zeta_2 + \bar{\zeta}_2\right) \\ & +\alpha_{21}\left(\zeta_1 + \bar{\zeta}_1\right)^3 + \alpha_{23}\left(\zeta_1 + \bar{\zeta}_1\right)\left(\zeta_2 + \bar{\zeta}_2\right)^2 + \alpha_{24}\left(\zeta_2 + \bar{\zeta}_2\right)^3\Big] \\ & + \frac{i}{2\omega}\left[F_{21}\left(\zeta_1 + \bar{\zeta}_1\right) + F_{22}\left(\zeta_2 + \bar{\zeta}_2\right)\right](z + \bar{z})\end{aligned} \tag{9.67}$$

Next, we determine the normal forms of (9.66) and (9.67) for weak damping and nonlinearities and two cases of parametric resonance: principal parametric resonance (i.e., $\Omega \approx 2\omega$) and fundamental parametric resonance (i.e., $\Omega \approx \omega$).

The Case of Principal Parametric Resonance

As a first step, we scale the variables. If we assume that $\zeta_1 = O(\epsilon)$, where ϵ is a small nondimensional parameter, then, due to the nonsemisimple structure of the linear operator, $\zeta_2 = O(\epsilon^{1-\lambda_2})$, where $\lambda_2 > 0$. Moreover, because the damping is weak, $\mu_n = O(\epsilon^{\lambda_1})$, where $\lambda_1 > 0$. Balancing the damping terms and the dominant nonlinear terms in (9.66) and (9.67), as done in the preceding section, we find that $\lambda_1 = \lambda_2 = 1/2$. For the case of principal parametric resonance, $\Omega \approx 2\omega$ and the terms $z\bar{\zeta}_1$ and $z\bar{\zeta}_2$ are near-resonance terms. In order that the damping and nonlinearity balance the dominant parametric resonance terms, $F_m = O(\epsilon)$. Thus, we introduce new variables defined by

$$\zeta_1 = \epsilon\eta_1\,, \quad \zeta_2 = \epsilon^{1/2}\eta_2\,, \quad \mu_n = \epsilon^{1/2}\hat{\mu}_n\,, \quad F_{mn} = \epsilon f_{mn}$$

into (9.66) and (9.67) and obtain

$$\dot{\eta}_1 = i\omega\eta_1 - \epsilon^{1/2}\hat{\mu}_1(\eta_1 - \bar{\eta}_1) + \frac{i\epsilon^{1/2}}{2\omega}\left[\alpha_{14}(\eta_2 + \bar{\eta}_2)^3 + f_{12}(\eta_2 + \bar{\eta}_2)(z + \bar{z})\right] + \cdots \tag{9.68}$$

$$\dot{\eta}_2 = i\omega\eta_2 - \epsilon^{1/2}\hat{\mu}_2(\eta_2 - \bar{\eta}_2) + \frac{i\epsilon^{1/2}}{2\omega}(\eta_1 + \bar{\eta}_1) + \cdots \tag{9.69}$$

To simplify (9.68) and (9.69), we use the near-identity transformation (9.60), choose the h_m to eliminate the nonresonance terms, and obtain

$$\dot{\xi}_1 = i\omega\xi_1 - \epsilon^{1/2}\hat{\mu}_1\xi_1 + \frac{3i\epsilon^{1/2}}{2\omega}\alpha_{14}\xi_2^2\bar{\xi}_2 + \frac{i\epsilon^{1/2}}{2\omega}f_{12}z\bar{\xi}_2 + \cdots \tag{9.70}$$

$$\dot{\xi}_2 = i\omega\xi_2 - \epsilon^{1/2}\hat{\mu}_2\xi_2 + \frac{i\epsilon^{1/2}}{2\omega}\xi_1 + \cdots \tag{9.71}$$

Equations 9.70 and 9.71 agree with those obtained by Tezak, Nayfeh, and Mook (1982) by using the method of multiple scales. Moreover, when $\hat{\mu}_n = 0$, (9.70) and (9.71) agree with those obtained by Namachchivaya and Malhotra (1992). In the absence of the parametric resonance, (9.70) and (9.71) reduce to (9.61) and (9.62). Furthermore, in the absence of the nonlinearity, (9.70) and (9.71) reduce to (7.86) and (7.87).

Fundamental Parametric Resonance

In this case, $\Omega \approx \omega$ and $z\bar{z}\zeta_1$ and $z\bar{z}\zeta_2$ are resonance terms and $z^2\bar{\zeta}_1$ and $z^2\bar{\zeta}_2$ are near-resonance terms. Hence, we scale F_{mn} to be $O(\epsilon^{1/2})$ so that the damping and nonlinear terms balance the parametric resonance terms. Therefore, we introduce new scaled variables defined by

$$\zeta_1 = \epsilon\eta_1\,, \quad \zeta_2 = \epsilon^{1/2}\eta_2\,, \quad \mu_n = \epsilon^{1/2}\hat{\mu}_n\,, \quad \text{and} \quad F_{mn} = \epsilon^{1/2}f_{mn}$$

into (9.66) and (9.67) and obtain

$$\dot{\eta}_1 = i\omega\eta_1 + \frac{i}{2\omega}f_{12}(\eta_2 + \bar{\eta}_2)(z + \bar{z}) - \epsilon^{1/2}\hat{\mu}_1(\eta_1 - \bar{\eta}_1) + \frac{i\epsilon^{1/2}}{2\omega}\left[\alpha_{14}(\eta_2 + \bar{\eta}_2)^3 + f_{11}(\eta_1 + \bar{\eta}_1)(z + \bar{z})\right] + \cdots \tag{9.72}$$

$$\dot{\eta}_2 = i\omega\eta_2 - \epsilon^{1/2}\hat{\mu}_2(\eta_2 - \bar{\eta}_2) + \frac{i\epsilon^{1/2}}{2\omega}\left[\eta_1 + \bar{\eta}_1 + f_{22}(\eta_2 + \bar{\eta}_2)(z + \bar{z})\right] + \cdots \tag{9.73}$$

To simplify (9.72) and (9.73), we introduce the transformation

$$\eta_1 = \xi_1 + h_{11}\left(\xi_n, \bar{\xi}_n, z, \bar{z}\right) + \epsilon^{1/2} h_{12}\left(\xi_n, \bar{\xi}_n, z, \bar{z}\right) + \cdots \tag{9.74}$$

$$\eta_2 = \xi_2 + \epsilon^{1/2} h_{22}\left(\xi_n, \bar{\xi}_n, z, \bar{z}\right) + \cdots \tag{9.75}$$

and choose the h_{mn} so that

$$\dot{\xi}_1 = i\omega\xi_1 + \epsilon^{1/2} g_1\left(\xi_n, \bar{\xi}_n, z, \bar{z}\right) + \cdots \tag{9.76}$$

$$\dot{\xi}_2 = i\omega\xi_2 + \epsilon^{1/2} g_2\left(\xi_n, \bar{\xi}_n, z, \bar{z}\right) + \cdots \tag{9.77}$$

have the simplest possible form. Substituting (9.74)–(9.77) into (9.72) and (9.73), using (9.65), and equating coefficients of like powers of ϵ, we obtain

$$\begin{aligned} & i\omega\left[\frac{\partial h_{11}}{\partial \xi_1}\xi_1 - \frac{\partial h_{11}}{\partial \bar{\xi}_1}\bar{\xi}_1 + \frac{\partial h_{11}}{\partial \xi_2}\xi_2 - \frac{\partial h_{11}}{\partial \bar{\xi}_2}\bar{\xi}_2 - h_{11}\right] + i\Omega\left(\frac{\partial h_{11}}{\partial z}z - \frac{\partial h_{11}}{\partial \bar{z}}\bar{z}\right) \\ & = \frac{i}{2\omega} f_{12}\left(\xi_2 + \bar{\xi}_2\right)(z + \bar{z}) \end{aligned} \tag{9.78}$$

$$\begin{aligned} & g_1 + i\omega\left[\frac{\partial h_{12}}{\partial \xi_1}\xi_1 - \frac{\partial h_{12}}{\partial \bar{\xi}_1}\bar{\xi}_1 + \frac{\partial h_{12}}{\partial \xi_2}\xi_2 - \frac{\partial h_{12}}{\partial \bar{\xi}_2}\bar{\xi}_2 - h_{12}\right] \\ & \quad + i\Omega\left(\frac{\partial h_{12}}{\partial z}z - \frac{\partial h_{12}}{\partial \bar{z}}\bar{z}\right) = -g_1\frac{\partial h_{11}}{\partial \xi_1} - \bar{g}_1\frac{\partial h_{11}}{\partial \bar{\xi}_1} - g_2\frac{\partial h_{11}}{\partial \xi_2} - \bar{g}_2\frac{\partial h_{11}}{\partial \bar{\xi}_2} \\ & \quad + \frac{i}{2\omega} f_{12}\left(h_{22} + \bar{h}_{22}\right)(z + \bar{z}) - \hat{\mu}_1\left(\xi_1 + h_{11} - \bar{\xi}_1 - \bar{h}_{11}\right) \\ & \quad + \frac{i}{2\omega}\left[\alpha_{14}\left(\xi_2 + \bar{\xi}_2\right)^3 + f_{11}\left(\xi_1 + \bar{\xi}_1 + h_{11} + \bar{h}_{11}\right)(z + \bar{z})\right] \end{aligned} \tag{9.79}$$

$$\begin{aligned} & g_2 + i\omega\left(\frac{\partial h_{22}}{\partial \xi_1}\xi_1 - \frac{\partial h_{22}}{\partial \bar{\xi}_1}\bar{\xi}_1 + \frac{\partial h_{22}}{\partial \xi_2}\xi_2 - \frac{\partial h_{22}}{\partial \bar{\xi}_2}\bar{\xi}_2 - h_{22}\right) \\ & \quad + i\Omega\left(\frac{\partial h_{22}}{\partial z}z - \frac{\partial h_{22}}{\partial \bar{z}}\bar{z}\right) = -\hat{\mu}_2\left(\xi_2 - \bar{\xi}_2\right) + \frac{i}{2\omega}\Big[\xi_1 + \bar{\xi}_1 \\ & \quad + h_{11} + \bar{h}_{11} + f_{22}\left(\xi_2 + \bar{\xi}_2\right)(z + \bar{z})\Big] \end{aligned} \tag{9.80}$$

The right-hand side of (9.78) suggests seeking h_{11} in the form

$$h_{11} = \Gamma_1\xi_2 z + \Gamma_2\xi_2\bar{z} + \Gamma_3\bar{\xi}_2 z + \Gamma_4\bar{\xi}_2\bar{z} \tag{9.81}$$

Substituting (9.81) into (9.78) and equating each of the coefficients of $\xi_2 z$, $\xi_2 \bar{z}$, $\bar{\xi}_2 z$, and $\bar{\xi}_2 \bar{z}$ on both sides, we obtain

$$\Gamma_1 = \frac{f_{12}}{2\omega\Omega}\,, \quad \Gamma_2 = -\frac{f_{12}}{2\omega\Omega}\,,$$
$$\Gamma_3 = \frac{f_{12}}{2\omega(\Omega - 2\omega)}\,, \quad \Gamma_4 = -\frac{f_{12}}{2\omega(\Omega + 2\omega)} \tag{9.82}$$

Substituting (9.81) into (9.80) and using (9.82) yields

$$\begin{aligned} g_2 &+ i\omega\left(\frac{\partial h_{22}}{\partial \xi_1}\xi_1 - \frac{\partial h_{22}}{\partial \bar{\xi}_1}\bar{\xi}_1 + \frac{\partial h_{22}}{\partial \xi_2}\xi_2 - \frac{\partial h_{22}}{\partial \bar{\xi}_2}\bar{\xi}_2 - h_{22}\right) \\ &+ i\Omega\left(\frac{\partial h_{22}}{\partial z}z - \frac{\partial h_{22}}{\partial \bar{z}}\bar{z}\right) = -\hat{\mu}_2\left(\xi_2 - \bar{\xi}_2\right) \\ &+ \frac{i}{2\omega}\left[\xi_1 + \bar{\xi}_1 + \frac{f_{12}}{\Omega(\Omega + 2\omega)}\left(\xi_2 z + \bar{\xi}_2 \bar{z}\right)\right. \\ &\left.+ \frac{f_{12}}{\Omega(\Omega - 2\omega)}\left(\xi_2 \bar{z} + \bar{\xi}_2 z\right) + f_{22}\left(\xi_2 + \bar{\xi}_2\right)(z + \bar{z})\right] \end{aligned} \tag{9.83}$$

Equating g_2 to the resonance and near-resonance terms on the right-hand side of (9.83), we have

$$g_2 = -\hat{\mu}_2 \xi_2 + \frac{i}{2\omega}\xi_1 \tag{9.84}$$

Then, we seek h_{22} in the form

$$h_{22} = \Gamma_5 \bar{\xi}_2 + \Gamma_6 \bar{\xi}_1 + \Gamma_7 \xi_2 z + \Gamma_8 \xi_2 \bar{z} + \Gamma_9 \bar{\xi}_2 z + \Gamma_{10} \bar{\xi}_2 \bar{z} \tag{9.85}$$

Substituting (9.85) into (9.83), using (9.84), and equating each of the coefficients of $\bar{\xi}_1$, $\bar{\xi}_2$, $\xi_2 z$, $\xi_2 \bar{z}$, $\bar{\xi}_2 z$, and $\bar{\xi}_2 \bar{z}$ on both sides, we obtain

$$\Gamma_5 = \frac{i\hat{\mu}_2}{2\omega}\,, \quad \Gamma_6 = -\frac{1}{4\omega^2}\,,$$
$$\Gamma_7 = \frac{f_{22}}{2\omega\Omega} + \frac{f_{12}}{2\omega\Omega^2(\Omega + 2\omega)}\,,$$
$$\Gamma_8 = -\frac{f_{22}}{2\omega\Omega} - \frac{f_{12}}{2\omega\Omega^2(\Omega - 2\omega)}\,,$$
$$\Gamma_9 = \frac{f_{22}}{2\omega(\Omega - 2\omega)} + \frac{f_{12}}{2\omega\Omega(\Omega - 2\omega)^2}\,,$$
$$\Gamma_{10} = -\frac{f_{22}}{2\omega(\Omega + 2\omega)} - \frac{f_{12}}{2\omega\Omega(\Omega + 2\omega)^2} \tag{9.86}$$

Substituting (9.81), (9.82), (9.85), and (9.86) into (9.79) and using (9.84), we obtain

$$\begin{aligned}
&g_1 + i\omega\left[\frac{h_{12}}{\partial \xi_1}\xi_1 - \frac{\partial h_{12}}{\partial \bar{\xi}_1}\bar{\xi}_1 + \frac{\partial h_{12}}{\partial \xi_2}\xi_2 - \frac{\partial h_{12}}{\partial \bar{\xi}_2}\bar{\xi}_2 - h_{12}\right] \\
&\quad + i\Omega\left(\frac{\partial h_{12}}{\partial z} - \frac{\partial h_{12}}{\partial \bar{z}}\bar{z}\right) \\
&= (\Gamma_1 z + \Gamma_2 \bar{z})\left(\hat{\mu}_2\xi_2 - \frac{i}{2\omega}\xi_1\right) + (\Gamma_3 z + \Gamma_4 \bar{z})\left(\hat{\mu}_2\bar{\xi}_2 + \frac{i}{2\omega}\bar{\xi}_1\right) \\
&\quad - \hat{\mu}_1\left(\xi_1 - \bar{\xi}_1\right) - \frac{\hat{\mu}_1 f_{12}(\Omega+\omega)}{\omega\Omega(\Omega+2\omega)}\left(\xi_2 z - \bar{\xi}_2\bar{z}\right) \\
&\quad + \frac{\hat{\mu}_1 f_{12}(\Omega-\omega)}{\omega\Omega(\Omega-2\omega)}\left(\xi_2\bar{z} - \bar{\xi}_2 z\right) \\
&\quad + \frac{i}{2\omega} f_{12}(z+\bar{z})\left[\frac{i\hat{\mu}_2}{2\omega}\left(\bar{\xi}_2 - \xi_2\right) + \left(\frac{f_{22}}{\Omega(\Omega+2\omega)} + \frac{f_{12}}{\Omega^2(\Omega+2\omega)^2}\right)\right. \\
&\quad \bullet\left(\xi_2 z + \bar{\xi}_2\bar{z}\right) - \frac{1}{4\omega^2}\left(\xi_1 + \bar{\xi}_1\right) + \left(\frac{f_{22}}{\Omega(\Omega-2\omega)} + \frac{f_{12}}{\Omega^2(\Omega-2\omega)^2}\right) \\
&\quad \left.\bullet\left(\xi_2\bar{z} + \bar{\xi}_2 z\right)\right] + \frac{i}{2\omega}\left[f_{11}\left(\xi_1 + \bar{\xi}_1\right)(z+\bar{z})\right. \\
&\quad + \frac{f_{11}f_{12}}{\Omega(\Omega+2\omega)}\left(\xi_2 z + \bar{\xi}_2\bar{z}\right)\bullet(z+\bar{z}) \\
&\quad \left.+ \frac{f_{11}f_{12}}{\Omega(\Omega-2\omega)}\left(\xi_2\bar{z} + \bar{\xi}_2 z\right)(z+\bar{z}) + \alpha_{14}\left(\xi_2 + \bar{\xi}_2\right)^3\right]
\end{aligned} \tag{9.87}$$

Because we are stopping at this order, we do not need to explicitly calculate h_{12}. Then, equating g_1 to the resonance and near-resonance terms in (9.87), we obtain

$$\begin{aligned}
g_1 &= -\hat{\mu}_1\xi_1 + \frac{3i}{2\omega}\alpha_{14}\xi_2^2\bar{\xi}_2 \\
&\quad + \frac{i}{2\omega}\left[\frac{f_{12}^2}{\Omega^2}\left(\frac{1}{(\Omega+2\omega)^2} + \frac{1}{(\Omega-2\omega)^2}\right) + \frac{2f_{12}(f_{11}+f_{22})}{\Omega^2 - 4\omega^2}\right] z\bar{z}\xi_2 \\
&\quad + \frac{i}{2\omega}\left[\frac{f_{12}(f_{11}+f_{22})}{\Omega(\Omega-2\omega)} + \frac{f_{12}^2}{\Omega^2(\Omega-2\omega)^2}\right] z^2\bar{\xi}_2
\end{aligned} \tag{9.88}$$

Finally, substituting (9.81) and (9.82) into (9.74) yields

$$\eta_1 = \xi_1 + \frac{f_{12}}{2\omega}\left[\frac{\xi_2 z}{\Omega} - \frac{\xi_2\bar{z}}{\Omega} + \frac{\bar{\xi}_2 z}{\Omega - 2\omega} - \frac{\bar{\xi}_2\bar{z}}{\Omega + 2\omega}\right] + \cdots \tag{9.89}$$

Then, substituting (9.88) and (9.84) into (9.76) and (9.77), we obtain the normal form

$$\dot{\xi}_1 = i\omega\xi_1 - \epsilon^{1/2}\hat{\mu}_1\xi_1 + \frac{i\epsilon^{1/2}}{2\omega}\left\{\left[\frac{f_{12}(f_{11}+f_{22})}{\Omega(\Omega-2\omega)} + \frac{f_{12}^2}{\Omega^2(\Omega-2\omega)^2}\right]z^2\bar{\xi}_2 + \left[\frac{2f_{12}(f_{11}+f_{22})}{\Omega^2-4\omega^2} + \frac{f_{12}^2}{\Omega^2(\Omega+2\omega)^2} + \frac{f_{12}^2}{\Omega^2(\Omega-2\omega)^2}\right]\xi_2 + 3\alpha_{14}\xi_2^2\bar{\xi}_2\right\} + \cdots \tag{9.90}$$

$$\dot{\xi}_2 = i\omega\xi_2 - \epsilon^{1/2}\hat{\mu}_2\xi_2 + \frac{i\epsilon^{1/2}}{2\omega}\xi_1 + \cdots \tag{9.91}$$

because $z\bar{z} = 1$.

In the absence of the parametric excitation (i.e., $f_{mn} = 0$), (9.90) and (9.91) reduce to (9.61) and (9.62). In the absence of the nonlinearity (i.e., $\alpha_{14} = 0$), (9.90) and (9.91) reduce to (7.134) and (7.135) when $f_{13} = 0$ and $f_{32} = 0$.

9.2 Gyroscopic Systems

Again, for simplicity, we consider a two-degree-of-freedom system to illustrate the method. Specifically, we consider

$$\ddot{u}_1 + \lambda_1\dot{u}_2 + \alpha_1 u_1 = \epsilon\left(4\delta_1 u_1^3 + 3\delta_2 u_1^2 u_2 + 2\delta_3 u_1 u_2^2 + \delta_4 u_2^3\right) + 2f_1\cos(\Omega t + \tau_1) \tag{9.92}$$

$$\ddot{u}_2 - \lambda_2\dot{u}_1 + \alpha_2 u_2 = \epsilon\left(\delta_2 u_1^3 + 2\delta_3 u_1^2 u_2 + 3\delta_4 u_1 u_2^2 + 4\delta_5 u_2^3\right) + 2f_2\cos(\Omega t + \tau_2) \tag{9.93}$$

Using the transformation (7.144)–(7.147) and following steps similar to those used in Section 7.4, we cast (9.92) and (9.93) in the complex-valued forms

$$\dot{\zeta}_1 = i\omega_1\zeta_1 + \frac{i\omega_1(\alpha_1-\omega_2^2)}{2\alpha_1(\omega_2^2-\omega_1^2)}(z_1+\bar{z}_1) - \frac{\lambda_1}{2(\omega_2^2-\omega_1^2)}(z_2+\bar{z}_2) + \epsilon\chi_{11}\left(\zeta_1+\bar{\zeta}_1+\zeta_2+\bar{\zeta}_2\right)^3 + \epsilon\chi_{12}\left(\zeta_1+\bar{\zeta}_1+\zeta_2+\bar{\zeta}_2\right)^2 \bullet \left[\frac{i(\alpha_1-\omega_1^2)}{\lambda_1\omega_1}(\zeta_1-\bar{\zeta}_1) + \frac{i(\alpha_1-\omega_2^2)}{\lambda_1\omega_2}(\zeta_2-\bar{\zeta}_2)\right] + \epsilon\chi_{14}\left[\frac{i(\alpha_1-\omega_1^2)}{\lambda_1\omega_1}(\zeta_1-\bar{\zeta}_1) + \frac{i(\alpha_1-\omega_2^2)}{\lambda_1\omega_2}(\zeta_2-\bar{\zeta}_2)\right]^3 + \epsilon\chi_{13}\left(\zeta_1+\bar{\zeta}_1+\zeta_2+\bar{\zeta}_2\right)\left[\frac{i(\alpha_1-\omega_1^2)}{\lambda_1\omega_1}(\zeta_1-\bar{\zeta}_1) + \frac{i(\alpha_1-\omega_2^2)}{\lambda_1\omega_2}(\zeta_2-\bar{\zeta}_2)\right]^2 \tag{9.94}$$

$$\dot{\zeta}_2 = i\omega_2\zeta_2 - \frac{i\omega_2(\alpha_1-\omega_1^2)}{2\alpha_1(\omega_2^2-\omega_1^2)}(z_1+\bar{z}_1) + \frac{\lambda_1}{2(\omega_2^2-\omega_1^2)}(z_2+\bar{z}_2)$$
$$+\epsilon\chi_{21}\left(\zeta_1+\bar{\zeta}_1+\zeta_2+\bar{\zeta}_2\right)^3+\epsilon\chi_{22}\left(\zeta_1+\bar{\zeta}_1+\zeta_2+\bar{\zeta}_2\right)^2$$
$$\bullet\left[\frac{i(\alpha_1-\omega_1^2)}{\lambda_1\omega_1}\left(\zeta_1-\bar{\zeta}_1\right)+\frac{i(\alpha_1-\omega_2^2)}{\lambda_1\omega_2}\left(\zeta_2-\bar{\zeta}_2\right)\right]$$
$$+\epsilon\chi_{24}\left[\frac{i(\alpha_1-\omega_1^2)}{\lambda_1\omega_1}\left(\zeta_1-\bar{\zeta}_1\right)+\frac{i(\alpha_1-\omega_2^2)}{\lambda_1\omega_2}\left(\zeta_2-\bar{\zeta}_2\right)\right]^3$$
$$+\epsilon\chi_{23}\left(\zeta_1+\bar{\zeta}_1+\zeta_2+\bar{\zeta}_2\right)\left[\frac{i(\alpha_1-\omega_1^2)}{\lambda_1\omega_1}\left(\zeta_1-\bar{\zeta}_1\right)\right.$$
$$\left.+\frac{i(\alpha_1-\omega_2^2)}{\lambda_1\omega_2}\left(\zeta_2-\bar{\zeta}_2\right)\right]^2 \tag{9.95}$$

where

$$\chi_{11}=\frac{4i\delta_1\omega_1(\alpha_1-\omega_2^2)-\alpha_1\delta_2\lambda_1}{2\alpha_1(\omega_2^2-\omega_1^2)},\quad \chi_{12}=\frac{3i\delta_2\omega_1(\alpha_1-\omega_2^2)-2\alpha_1\delta_3\lambda_1}{2\alpha_1(\omega_2^2-\omega_1^2)},$$
$$\chi_{13}=\frac{2i\delta_3\omega_1(\alpha_1-\omega_2^2)-3\alpha_1\delta_4\lambda_1}{2\alpha_1(\omega_2^2-\omega_1^2)},\quad \chi_{14}=\frac{i\delta_4\omega_1(\alpha_1-\omega_2^2)-4\alpha_1\delta_5\lambda_1}{2\alpha_1(\omega_2^2-\omega_1^2)},$$
$$\chi_{21}=\frac{\alpha_1\delta_2\lambda_2-4i\delta_1\omega_2(\alpha_1-\omega_1^2)}{2\alpha_1(\omega_2^2-\omega_1^2)},\quad \chi_{22}=\frac{2\alpha_1\delta_3\lambda_1-3i\delta_2\omega_2(\alpha_1-\omega_1^2)}{2\alpha_1(\omega_2^2-\omega_1^2)},$$
$$\chi_{23}=\frac{3\alpha_1\delta_4\lambda_1-2i\delta_3\omega_2(\alpha_1-\omega_1^2)}{2\alpha_1(\omega_2^2-\omega_1^2)},\quad \chi_{24}=\frac{4\alpha_1\delta_5\lambda_1-i\delta_4\omega_2(\alpha_1-\omega_1^2)}{2\alpha_1(\omega_2^2-\omega_1^2)} \tag{9.96}$$

There are two cases to consider: $\Omega \approx \omega_m$ (corresponding to the primary resonance of the mth mode) and Ω is away from ω_1 and ω_2. The case of primary resonance is discussed next and the other case (secondary resonances) is discussed in Section 9.2.2.

9.2.1 Primary Resonances

As in Section 8.2.1, we treat this case by scaling z_1 and z_2 at $O(\epsilon)$. Then, substituting the near-identity transformation (8.60) into (9.94) and (9.95) and choosing the h_m to eliminate the nonresonance terms, we obtain the following normal forms:

No Internal Resonance, $\Omega \approx \omega_n$

$$\dot{\xi}_1 = i\omega_1\xi_1+\epsilon\left(S_{11}\xi_1^2\bar{\xi}_1+S_{12}\xi_2\bar{\xi}_2\xi_1\right)+\epsilon\Gamma_1\delta_{1n} \tag{9.97}$$

$$\dot{\xi}_2 = i\omega_2\xi_2+\epsilon\left(S_{21}\xi_1\bar{\xi}_1\xi_2+S_{22}\xi_2^2\bar{\xi}_2\right)+\epsilon\Gamma_2\delta_{2n} \tag{9.98}$$

where

$$S_{11} = 3\chi_{11} + \frac{i\chi_{12}(\alpha_1 - \omega_1^2)}{\lambda_1\omega_1} + \frac{\chi_{13}(\alpha_1 - \omega_1^2)^2}{\lambda_1^2\omega_1^2} + \frac{3i\chi_{14}(\alpha_1 - \omega_1^2)^3}{\lambda_1^3\omega_1^3} \quad (9.99)$$

$$S_{12} = 6\chi_{11} + \frac{2i\chi_{12}(\alpha_1 - \omega_1^2)}{\lambda_1\omega_1} + \frac{2\chi_{13}(\alpha_1 - \omega_2^2)^2}{\lambda_1^2\omega_2^2} + \frac{6i\chi_{14}(\alpha_1 - \omega_1^2)(\alpha_1 - \omega_2^2)^2}{\lambda_1^3\omega_1\omega_2^2} \quad (9.100)$$

$$S_{21} = 6\chi_{21} + \frac{2i\chi_{22}(\alpha_1 - \omega_2^2)}{\lambda_1\omega_2} + \frac{2\chi_{23}(\alpha_1 - \omega_1^2)^2}{\lambda_1^2\omega_1^2} + \frac{6i\chi_{24}(\alpha_1 - \omega_1^2)^2(\alpha_1 - \omega_2^2)}{\lambda_1^3\omega_1^2\omega_2} \quad (9.101)$$

$$S_{22} = 3\chi_{21} + \frac{i\chi_{22}(\alpha_1 - \omega_2^2)}{\lambda_1\omega_2} + \frac{\chi_{23}(\alpha_1 - \omega_2^2)^2}{\lambda_1^2\omega_2^2} + \frac{3i\chi_{24}(\alpha_1 - \omega_2^2)^3}{\lambda_1^3\omega_2^3} \quad (9.102)$$

$$\Gamma_1 = \frac{i\omega_1(\alpha_2 - \omega_1^2)z_1 - \lambda_1\alpha_1 z_2}{2\alpha_1(\omega_2^2 - \omega_1^2)}, \quad \Gamma_2 = -\frac{i\omega_2(\alpha_1 - \omega_1^2)z_1 - \lambda_1\alpha_1 z_2}{2\alpha_1(\omega_2^2 - \omega_1^2)} \quad (9.103)$$

Three-to-One Internal Resonance, $\Omega \approx \omega_n$

$$\dot{\xi}_1 = i\omega_1\xi_1 + \epsilon\left(S_{11}\xi_1^2\bar{\xi}_1 + S_{12}\xi_2\bar{\xi}_2\xi_1 + S_{13}\xi_2\bar{\xi}_1^2\right) + \epsilon\Gamma_1\delta_{1n} \quad (9.104)$$

$$\dot{\xi}_2 = i\omega_2\xi_2 + \epsilon\left(S_{21}\xi_1\bar{\xi}_1\xi_2 + S_{22}\xi_2^2\bar{\xi}_2 + S_{23}\xi_1^3\right) + \epsilon\Gamma_2\delta_{2n} \quad (9.105)$$

where S_{11}, S_{12}, S_{21}, S_{22}, Γ_1, and Γ_2 are defined in (9.99)–(9.103) and

$$S_{13} = 3\chi_{11} + \frac{2i\chi_{12}}{\lambda_1}\left[\frac{\alpha_1 - \omega_2^2}{2\omega_2} - \frac{\alpha_1 - \omega_1^2}{\omega_1}\right] - \frac{3i\chi_{14}(\alpha_1 - \omega_1^2)^2(\alpha_1 - \omega_2^2)}{\lambda_1^3\omega_1^2\omega_2}$$
$$+ \frac{2\chi_{13}(\alpha_1 - \omega_1^2)}{\lambda^2\omega_1}\left[\frac{\alpha_1 - \omega_2^2}{\omega_2} - \frac{\alpha_1 - \omega_1^2}{2\omega_1}\right] \quad (9.106)$$

$$S_{23} = \chi_{21} + \frac{i\chi_{22}(\alpha_1 - \omega_1^2)}{\lambda_1\omega_1} - \frac{\chi_{23}(\alpha_1 - \omega_1^2)^2}{\lambda_1^2\omega_1^2} - \frac{i\chi_{24}(\alpha_1 - \omega_1^2)^3}{\lambda_1^3\omega_1^3} \quad (9.107)$$

We note that the transformation (7.144)–(7.147) is not valid when $\omega_2 \approx \omega_1$ and hence (9.94) and (9.95) do not apply in this case. Consequently, the case of a one-to-one internal resonance needs special treatment.

9.2.2 Secondary Resonances in the Absence of Internal Resonances

In this case, we scale z_1 and z_2 at $O(1)$. Then, we introduce the transformation (8.81) and (8.82) into (9.94) and (9.95) to eliminate the terms involving z_n and $\bar{z}_n$

at $O(1)$ and obtain

$$\begin{aligned}\dot{\eta}_1 = {} & i\omega_1\eta_1 + \epsilon\chi_{11}\left[\eta_1 + \bar{\eta}_1 + \eta_2 + \bar{\eta}_2 + b_1(z_1 + \bar{z}_1) + b_2(z_2 - \bar{z}_2)\right]^3 \\ & + \epsilon\chi_{12}\left[\eta_1 + \bar{\eta}_1 + \eta_2 + \bar{\eta}_2 + b_1(z_1 + \bar{z}_1) + b_2(z_2 - \bar{z}_2)\right]^2 \\ & \bullet\left[\frac{i(\alpha_1 - \omega_1^2)}{\lambda_1\omega_1}(\eta_1 - \bar{\eta}_1) + \frac{i(\alpha_1 - \omega_2^2)}{\lambda_1\omega_2}(\eta_2 - \bar{\eta}_2)\right. \\ & \left. + b_3(z_1 - \bar{z}_1) + b_4(z_2 + \bar{z}_2)\right] \\ & + \epsilon\chi_{13}\left[\eta_1 + \bar{\eta}_1 + \eta_2 + \bar{\eta}_2 + b_1(z_1 + \bar{z}_1) + b_2(z_2 - \bar{z}_2)\right] \\ & \bullet\left[\frac{i(\alpha_1 - \omega_1^2)}{\lambda_1\omega_1}(\eta_1 - \bar{\eta}_1) + \frac{i(\alpha_1 - \omega_2^2)}{\lambda_1\omega_2}(\eta_2 - \bar{\eta}_2)\right. \\ & \left. + b_3(z_1 - \bar{z}_1) + b_4(z_2 + \bar{z}_2)\right]^2 \\ & + \epsilon\chi_{14}\left[\frac{i(\alpha_1 - \omega_1^2)}{\lambda_1\omega_1}(\eta_1 - \bar{\eta}_1) + \frac{i(\alpha_1 - \omega_2^2)}{\lambda_1\omega_2}(\eta_2 - \bar{\eta}_2)\right. \\ & \left. + b_3(z_1 - \bar{z}_1) + b_4(z_2 + \bar{z}_2)\right]^3 \end{aligned} \tag{9.108}$$

$$\begin{aligned}\dot{\eta}_2 = {} & i\omega_2\eta_2 + \epsilon\chi_{21}\left[\eta_1 + \bar{\eta}_1 + \eta_2 + \bar{\eta}_2 + b_1(z_1 + \bar{z}_1) + b_2(z_2 - \bar{z}_2)\right]^3 \\ & + \epsilon\chi_{22}\left[\eta_1 + \bar{\eta}_1 + \eta_2 + \bar{\eta}_2 + b_1(z_1 + \bar{z}_1) + b_2(z_2 - \bar{z}_2)\right]^2 \\ & \bullet\left[\frac{i(\alpha_1 - \omega_1^2)}{\lambda_1\omega_1}(\eta_1 - \bar{\eta}_1) + \frac{i(\alpha_1 - \omega_2^2)}{\lambda_1\omega_2}(\eta_2 - \bar{\eta}_2)\right. \\ & \left. + b_3(z_1 - \bar{z}_1) + b_4(z_2 + \bar{z}_2)\right] \\ & + \epsilon\chi_{23}\left[\eta_1 + \bar{\eta}_1 + \eta_2 + \bar{\eta}_2 + b_1(z_1 + \bar{z}_1) + b_2(z_2 - \bar{z}_2)\right] \\ & \bullet\left[\frac{i(\alpha_1 - \omega_1^2)}{\lambda_1\omega_1}(\eta_1 - \bar{\eta}_1) + \frac{i(\alpha_1 - \omega_2^2)}{\lambda_1\omega_2}(\eta_2 - \bar{\eta}_2)\right. \\ & \left. + b_3(z_1 - \bar{z}_1) + b_4(z_2 + \bar{z}_2)\right]^2 \\ & + \epsilon\chi_{24}\left[\frac{i(\alpha_1 - \omega_1^2)}{\lambda_1\omega_1}(\eta_1 - \bar{\eta}_1) + \frac{i(\alpha_1 - \omega_2^2)}{\lambda_1\omega_2}(\eta_2 - \bar{\eta}_2)\right. \\ & \left. + b_3(z_1 - \bar{z}_1) + b_4(z_2 + \bar{z}_2)\right]^3 \end{aligned} \tag{9.109}$$

where the b_i are defined in (8.85).

It follows from (9.108) and (9.109) that, to $O(\epsilon)$, only a three-to-one internal resonance is possible because (8.81) and (8.82) are not valid when $\omega_2 \approx \omega_1$ (i.e., when there is a one-to-one internal resonance). Moreover, the excitation can produce resonances corresponding to

- $\Omega \approx 3\omega_m$: Subharmonic resonance of order one-third,
- $\Omega \approx 1/3\omega_m$: Superharmonic resonance of order three,
- $\Omega \approx 2\omega_n \pm \omega_m$: Combination resonance.

We present the normal forms of (9.108) and (9.109) for several of these external resonances in the absence of internal resonances in this section and in the presence of a three-to-one internal resonance in Section 9.2.3.
Substituting the near-identity transformation (8.14) into (9.108) and (9.109) and choosing the h_m to eliminate the nonresonance terms, we obtain the following normal forms:

No External Resonance

$$\dot{\xi}_1 = i\omega_1\xi_1 + \epsilon\sigma_1\xi_1 + \epsilon\left(S_{11}\xi_1^2\bar{\xi}_1 + S_{12}\xi_2\bar{\xi}_2\xi_1\right) \tag{9.110}$$

$$\dot{\xi}_2 = i\omega_2\xi_2 + \epsilon\sigma_2\xi_2 + \epsilon\left(S_{21}\xi_1\bar{\xi}_1\xi_2 + S_{22}\xi_2^2\bar{\xi}_2\right) \tag{9.111}$$

where S_{11}, S_{12}, S_{21}, and S_{22} are defined in (9.99)–(9.102), and

$$\begin{aligned}
\sigma_1 = {} & 6\chi_{11}\left[b_1^2 z_1\bar{z}_1 - b_2^2 z_2\bar{z}_2 + b_1 b_2\left(z_2\bar{z}_1 - z_1\bar{z}_2\right)\right] \\
& + 2\chi_{12}\left[\frac{i(\alpha_1-\omega_1^2)}{\lambda_1\omega_1}\left(b_1^2 z_1\bar{z}_1 - b_2^2 z_2\bar{z}_2 + b_1 b_2\left(z_2\bar{z}_1 - z_1\bar{z}_2\right)\right)\right. \\
& \left. + \left(b_1 b_4 - b_2 b_3\right)\left(z_1\bar{z}_2 + \bar{z}_1 z_2\right)\right] \\
& + 2\chi_{13}\left[b_4^2 z_2\bar{z}_2 - b_3^2 z_1\bar{z}_1 + b_3 b_4\left(z_1\bar{z}_2 - \bar{z}_1 z_2\right)\right. \\
& \left. + \frac{i(\alpha_1-\omega_1^2)}{\lambda_1\omega_1}\left(b_1 b_4 - b_2 b_3\right)\left(z_1\bar{z}_2 + \bar{z}_1 z_2\right)\right] \\
& + \frac{6i\chi_{14}(\alpha_1-\omega_1^2)}{\lambda_1\omega_1}\left[b_4^2 z_2\bar{z}_2 - b_3^2 z_1\bar{z}_1 + b_3 b_4\left(z_1\bar{z}_2 - \bar{z}_1 z_2\right)\right]
\end{aligned} \tag{9.112}$$

$$\begin{aligned}
\sigma_2 = {} & 6\chi_{21}\left[b_1^2 z_1\bar{z}_1 - b_2^2 z_2\bar{z}_2 + b_1 b_2\left(z_2\bar{z}_1 - z_1\bar{z}_2\right)\right] \\
& + 2\chi_{32}\left[\frac{i(\alpha_1-\omega_1^2)}{\lambda_1\omega_1}\left(b_1^2 z_1\bar{z}_1 - b_2^2 z_2\bar{z}_2 + b_1 b_2\left(z_2\bar{z}_1 - z_1\bar{z}_2\right)\right)\right. \\
& \left. + \left(b_1 b_4 - b_2 b_3\right)\left(z_1\bar{z}_2 + \bar{z}_1 z_2\right)\right] \\
& + 2\chi_{23}\left[b_4^2 z_2\bar{z}_2 - b_3^2 z_1\bar{z}_1 + b_3 b_4\left(z_1\bar{z}_2 - \bar{z}_1 z_2\right)\right. \\
& \left. + \frac{i(\alpha_1-\omega_1^2)}{\lambda_1\omega_2}\left(b_1 b_4 - b_2 b_4\right)\left(z_1\bar{z}_2 + \bar{z}_1 z_2\right)\right] \\
& + \frac{6i\chi_{24}(\alpha_1-\omega_1^2)}{\lambda_1\omega_2}\left[b_4^2 z_2\bar{z}_2 - b_3^2 z_1\bar{z}_1 + b_3 b_4\left(z_1\bar{z}_2 - \bar{z}_1 z_2\right)\right]
\end{aligned} \tag{9.113}$$

$\Omega \approx 3\omega_2$

$$\dot{\xi}_1 = i\omega_1\xi_1 + \epsilon\sigma_1\xi_1 + \epsilon\left(S_{11}\xi_1^2\bar{\xi}_1 + S_{12}\xi_2\bar{\xi}_2\xi_1\right) \tag{9.114}$$

$$\dot{\xi}_2 = i\omega_2\xi_2 + \epsilon\sigma_2\xi_2 + \epsilon\left(S_{21}\xi_1\bar{\xi}_1\xi_2 + S_{22}\xi_2^2\bar{\xi}_2\right) + \epsilon\Gamma_1\bar{\xi}_2^2 \tag{9.115}$$

where σ_1 and σ_2 are defined in (9.112) and (9.113), the S_{mn} are defined in (9.99)–(9.102), and

$$\begin{aligned}\Gamma_1 = {} & (b_1z_1 + b_2z_2)\left[3\chi_{21} - \frac{2i\chi_{22}(\alpha_1 - \omega_2^2)}{\lambda_1\omega_2} - \frac{\chi_{23}(\alpha_1 - \omega_2^2)^2}{\lambda_1^2\omega_2^2}\right] \\ & + (b_3z_1 + b_4z_4)\left[\chi_{22} - \frac{2i\chi_{23}(\alpha_1 - \omega_2^2)}{\lambda_1\omega_2} - \frac{3\chi_{24}(\alpha_1 - \omega_2^2)^2}{\lambda_1^2\omega_2^2}\right]\end{aligned} \tag{9.116}$$

$\Omega \approx 1/3\omega_1$

$$\dot{\xi}_1 = i\omega_1\xi_1 + \epsilon\sigma_1\xi_1 + \epsilon\left(S_{11}\xi_1^2\bar{\xi}_1 + S_{12}\xi_2\bar{\xi}_2\xi_1\right) + \epsilon\Gamma_2 \tag{9.117}$$

$$\dot{\xi}_2 = i\omega_2\xi_2 + \epsilon\sigma_2\xi_2 + \epsilon\left(S_{21}\xi_1\bar{\xi}_1\xi_2 + S_{22}\xi_2^2\bar{\xi}_2\right) \tag{9.118}$$

where σ_1 and σ_2 are defined in (9.112) and (9.113), the S_{mn} are defined in (9.99)–(9.102), and

$$\begin{aligned}\Gamma_2 = {} & \chi_{11}(b_1z_1 + b_2z_2)^3 + \chi_{12}(b_1z_1 + b_2z_2)^2(b_3z_1 + b_4z_2) \\ & + \chi_{13}(b_1z_1 + b_2z_2)(b_3z_1 + b_4z_2)^2 + \chi_{14}(b_3z_1 + b_4z_2)^3\end{aligned} \tag{9.119}$$

$\Omega \approx \omega_2 + 2\omega_1$

$$\dot{\xi}_1 = i\omega_1\xi_1 + \epsilon\sigma_1\xi_1 + \epsilon\left(S_{11}\xi_1^2\bar{\xi}_1 + S_{12}\xi_2\bar{\xi}_2\xi_1\right) + \epsilon\Gamma_3\bar{\xi}_2\bar{\xi}_1 \tag{9.120}$$

$$\dot{\xi}_2 = i\omega_2\xi_2 + \epsilon\sigma_2\xi_2 + \epsilon\left(S_{21}\xi_1\bar{\xi}_1\xi_2 + S_{22}\xi_2^2\bar{\xi}_2\right) + \epsilon\Gamma_4\bar{\xi}_1^2 \tag{9.121}$$

where σ_1 and σ_2 are defined in (9.112) and (9.113), the S_{mn} are defined in (9.99)–(9.102), and

$$\begin{aligned}\Gamma_3 = {} & 6\chi_{11}(b_1z_1 + b_2z_2) + \frac{6\alpha_1\lambda_2}{\lambda_1\omega_1\omega_2}(b_3z_1 + b_4z_2) \\ & + 2\chi_{12}\left[b_3z_1 + b_4z_2 - i(b_1z_1 + b_2z_2)\left(\frac{\alpha_1 - \omega_1^2}{\lambda_1\omega_1} + \frac{\alpha_1 - \omega_2^2}{\lambda_1\omega_2}\right)\right] \\ & - 2\chi_{13}\left[i(b_3z_1 + b_4z_2)\left(\frac{\alpha_1 - \omega_1^2}{\lambda_1\omega_1} + \frac{\alpha_1 - \omega_2^2}{\lambda_1\omega_2}\right)\right. \\ & \left. - \frac{\alpha_1\lambda_2}{\lambda_1\omega_1\omega_2}(b_1z_1 + b_2z_2)\right]\end{aligned} \tag{9.122}$$

$$\Gamma_4 = 3\chi_{21}\left(b_1 z_1 + b_2 z_2\right) - 3\chi_{24}\frac{(\alpha_1 - \omega_1^2)^2}{\lambda_1^2\omega_1^2}\left(b_3 z_1 + b_4 z_2\right)$$
$$+ \chi_{22}\left[b_3 z_1 + b_4 z_2 - \frac{2i(\alpha_1 - \omega_1^2)}{\lambda_1\omega_1}\left(b_1 z_1 + b_2 z_2\right)\right]$$
$$- \chi_{23}\left[\frac{(\alpha_1 - \omega_1^2)^2}{\lambda_1^2\omega_1^2}\left(b_1 z_1 + b_2 z_2\right) + \frac{2i(\alpha_1 - \omega_1^2)}{\lambda_1\omega_1}\left(b_3 z_1 + b_4 z_2\right)\right] \tag{9.123}$$

9.2.3 Three-to-One Internal Resonance

Substituting the near-identity transformation (8.14) into (9.108) and (9.109) and choosing the h_m to eliminate the nonresonance terms, we obtain the following normal form in the absence of external resonances:

$$\dot{\xi}_1 = i\omega_1\xi_1 + \epsilon\sigma_1\xi_1 + \epsilon\left(S_{11}\xi_1^2\bar{\xi}_1 + S_{12}\xi_2\bar{\xi}_2\xi_1 + S_{13}\xi_2\bar{\xi}_1^2\right) \tag{9.124}$$

$$\dot{\xi}_2 = i\omega_2\xi_2 + \epsilon\sigma_2\xi_2 + \epsilon\left(S_{21}\xi_1\bar{\xi}_1\xi_2 + S_{22}\xi_2^2\bar{\xi}_2 + S_{23}\xi_1^3\right) \tag{9.125}$$

where σ_1 and σ_2 are defined in (9.112) and (9.113) and the S_{mn} are defined in (9.99)–(9.102), (9.106), and (9.107).

The normal forms of (9.108) and (9.109) in the presence of external resonances can be obtained by simply adding the appropriate Γ_m terms to (9.124) and (9.125), as in Section 9.2.2.

10
Systems with Quadratic and Cubic Nonlinearities

We consider the response of two-degree-of-freedom damped systems with quadratic and cubic nonlinearities to simultaneous principal and combination parametric resonances in the presence and absence of internal resonances. The case of no internal resonance is treated in Section 10.2, the case of three-to-one internal resonance is treated in Section 10.3, and the case of one-to-one internal resonance is treated in Section 10.4. The case of two-to-one internal resonance is treated in Section 10.5 using the method of normal forms, in Section 10.6 using the method of multiple scales, and in Section 10.7 using the generalized method of averaging. Finally, the case of a nonsemisimple one-to-one internal resonance is treated in Section 10.8.

10.1
Introduction

In this chapter, we consider the response of two-degree-of-freedom damped systems with quadratic and cubic nonlinearities to multifrequency parametric excitations. Specifically, we consider systems modeled by

$$\ddot{u}_1 + \omega_1^2 u_1 + 2\mu_1 \dot{u}_1 + \frac{\partial V}{\partial u_1}(u_1, u_2) + 2u_1 \sum_{m=1}^{M} f_{1m} \cos(\Omega_m t + \tau_{1m})$$
$$+ 2u_2 \sum_{m=1}^{m} f_{2m} \cos(\Omega_m t + \tau_{2m}) = 0 \qquad (10.1)$$

$$\ddot{u}_2 + \omega_2^2 u_2 + 2\mu_2 \dot{u}_2 + \frac{\partial V}{\partial u_2}(u_1, u_2) + 2u_1 \sum_{m=1}^{M} g_{1m} \cos(\Omega_m t + \nu_{1m})$$
$$+ 2u_2 \sum_{m=1}^{m} g_{2m} \cos(\Omega_m t + \nu_{2m}) = 0 \qquad (10.2)$$

$$V = \tfrac{1}{3}\delta_1 u_1^3 + \delta_2 u_1^2 u_2 + \delta_3 u_1 u_2^2 + \tfrac{1}{3}\delta_4 u_2^3 + \tfrac{1}{4}\alpha_1 u_1^4 + \alpha_2 u_1^3 u_2$$
$$+ \tfrac{1}{2}\alpha_3 u_1^2 u_2^2 + \alpha_4 u_1 u_2^3 + \tfrac{1}{4}\alpha_5 u_2^4 \qquad (10.3)$$

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Clearly, the undamped and unforced system is derivable from the Lagrangian

$$L = \tfrac{1}{2}\left(\dot{u}_1^2 + \dot{u}_2^2 + \omega_1^2 u_1^2 + \omega_2^2 u_2^2\right) - V(u_1, u_2)$$

Thus the undamped unforced normal form should be derivable from a Lagrangian, which can be used to check the accuracy of the calculated normal forms.

Again, as a first step, it is convenient to cast (10.1) and (10.2) in complex-valued form using the transformation

$$u_m = \zeta_m + \bar{\zeta}_m, \quad \dot{u}_m = i\omega_m\left(\zeta_m - \bar{\zeta}_m\right) \tag{10.4}$$

$$z_m = e^{i\Omega_m t}, \quad \dot{z}_m = i\Omega_m z_m \tag{10.5}$$

To keep track of the different orders of magnitude, we introduce a small nondimensional parameter ϵ. Then, if $u_m = O(\epsilon)$, $u_m^2 = O(\epsilon^2)$ and $u_m^3 = O(\epsilon^3)$, and we order the damping and excitation terms such that their effects balance each other as well as the effect of the nonlinearity. To reduce the amount of algebra, we assume that the μ_m, f_{mn}, and g_{mn} are $O(\epsilon^2)$. Using this ordering and the transformation (10.4) and (10.5), we rewrite (10.1)–(10.3) as

$$\begin{aligned}
\dot{\zeta}_1 = {} & i\omega_1\zeta_1 - \epsilon^2\mu_1\left(\zeta_1 - \bar{\zeta}_1\right) + \frac{i\epsilon}{2\omega_1}\left[\delta_1(\zeta_1 + \bar{\zeta}_1)^2\right.\\
& \left.+2\delta_2(\zeta_1 + \bar{\zeta}_1)(\zeta_2 + \bar{\zeta}_2) + \delta_3(\zeta_2 + \bar{\zeta}_2)^2\right] + \frac{i\epsilon^2}{2\omega_1}\left[\alpha_1(\zeta_1 + \bar{\zeta}_1)^3\right.\\
& \left.+3\alpha_2(\zeta_1 + \bar{\zeta}_1)^2(\zeta_2 + \bar{\zeta}_2) + \alpha_3(\zeta_1 + \bar{\zeta}_1)(\zeta_2 + \bar{\zeta}_2)^2 + \alpha_4(\zeta_2 + \bar{\zeta}_2)^3\right]\\
& + \frac{i\epsilon^2}{2\omega_1}(\zeta_1 + \bar{\zeta}_1)\sum_{m=1}^{M} f_{1m}\left(z_m e^{i\tau_{1m}} + \bar{z}_m e^{-i\tau_{1m}}\right)\\
& + \frac{i\epsilon^2}{2\omega_1}(\zeta_2 + \bar{\zeta}_2)\sum_{m=1}^{M} f_{2m}\left(z_m e^{i\tau_{2m}} + \bar{z}_m e^{-i\tau_{2m}}\right)
\end{aligned} \tag{10.6}$$

$$\begin{aligned}
\dot{\zeta}_2 = {} & i\omega_2\zeta_2 - \epsilon^2\mu_2\left(\zeta_2 - \bar{\zeta}_2\right) + \frac{i\epsilon}{2\omega_2}\left[\delta_2(\zeta_1 + \bar{\zeta}_1)^2\right.\\
& \left.+2\delta_3(\zeta_1 + \bar{\zeta}_1)(\zeta_2 + \bar{\zeta}_2) + \delta_4(\zeta_2 + \bar{\zeta})^2\right] + \frac{i\epsilon^2}{2\omega_2}\left[\alpha_2(\zeta_1 + \bar{\zeta}_1)^3\right.\\
& \left.+\alpha_3(\zeta_1 + \bar{\zeta}_1)^2(\zeta_2 + \bar{\zeta}_2) + 3\alpha_4(\zeta_1 + \bar{\zeta}_1)(\zeta_2 + \bar{\zeta}_2)^2 + \alpha_5(\zeta_2 + \bar{\zeta}_2)^3\right]\\
& + \frac{i\epsilon^2}{2\omega_2}(\zeta_1 + \bar{\zeta}_1)\sum_{m=1}^{M} g_{1m}\left(z_m e^{i\nu_{1m}} + \bar{z}_m e^{-i\nu_{1m}}\right)\\
& + \frac{i\epsilon^2}{2\omega_1}(\zeta_2 + \bar{\zeta}_2)\sum_{m=1}^{M} g_{2m}\left(z_m e^{i\nu_{2m}} + \bar{z}_m e^{-i\nu_{2m}}\right)
\end{aligned} \tag{10.7}$$

Next, we introduce the near-identity transformation

$$\zeta_m = \eta_m + \epsilon h_{m1}(\eta_n, \bar{\eta}_n) + \epsilon^2 h_{m2}(\eta_n, \bar{\eta}_n, z_n, \bar{z}_n) + \cdots \tag{10.8}$$

into (10.6) and (10.7) and choose the h_{mn} to eliminate the nonresonance terms so that the resulting equations have the simplest possible form

$$\dot{\eta}_m = i\omega_m \eta_m + \epsilon F_{m1}(\eta_n, \bar{\eta}_n) + \epsilon^2 F_{m2}(\eta_n, \bar{\eta}_n, z_n, \bar{z}_n) + \cdots \tag{10.9}$$

where the F_{mn} are chosen to eliminate the resonance and near-resonance terms. Substituting (10.8) and (10.9) into (10.6) and (10.7), using (10.5), and equating coefficients of like powers of ϵ, we obtain the following:

Order (ϵ)

$$\begin{aligned} F_{11} + \mathcal{L}(h_{11}) - i\omega_1 h_{11} = {} & \frac{i}{2\omega_1}\left[\delta_1(\eta_1+\bar{\eta}_1)^2 + 2\delta_2(\eta_1+\bar{\eta}_1)(\eta_2+\bar{\eta}_2)\right. \\ & \left. +\delta_3(\eta_2+\bar{\eta}_2)^2\right] \end{aligned} \tag{10.10}$$

$$\begin{aligned} F_{21} + \mathcal{L}(h_{21}) - i\omega_2 h_{21} = {} & \frac{i}{2\omega_2}\left[\delta_2(\eta_1+\bar{\eta}_1)^2 + 2\delta_3(\eta_1+\bar{\eta}_1)(\eta_2+\bar{\eta}_2)\right. \\ & \left. +\delta_4(\eta_2+\bar{\eta}_2)^2\right] \end{aligned} \tag{10.11}$$

Order (ϵ^2)

$$\begin{aligned} F_{12} + \mathcal{L}(h_{12}) - i\omega_1 h_{12} = {} & -\frac{\partial h_{11}}{\partial \eta_1}F_{11} - \frac{\partial h_{11}}{\partial \bar{\eta}_1}\bar{F}_{11} - \frac{\partial h_{11}}{\partial \eta_2}F_{21} - \frac{\partial h_{11}}{\partial \bar{\eta}_2}\bar{F}_{21} \\ & + \frac{i}{2\omega_1}\Big[2\delta_1(\eta_1+\bar{\eta}_1)(h_{11}+\bar{h}_{11}) + 2\delta_2(\eta_1+\bar{\eta}_1)(h_{21}+\bar{h}_{21}) \\ & +2\delta_2(h_{11}+\bar{h}_{11})(\eta_2+\bar{\eta}_2) + 2\delta_3(\eta_2+\bar{\eta}_2)(h_{21}+\bar{h}_{21})\Big] \\ & + \frac{i}{2\omega_1}\left[\alpha_1(\eta_1+\bar{\eta}_1)^3 + 3\alpha_2(\eta_1+\bar{\eta}_1)^2(\eta_2+\bar{\eta}_2)\right. \\ & \left. +\alpha_3(\eta_1+\bar{\eta}_1)(\eta_2+\bar{\eta}_2)^2 + \alpha_4(\eta_2+\bar{\eta}_2)^3\right] - \mu_1(\eta_1-\bar{\eta}_1) \\ & + \frac{i}{2\omega_1}(\eta_1+\bar{\eta}_1)\sum_{m=1}^{M} f_{1m}\left(z_m e^{i\tau_{1m}} + \bar{z}_m e^{-i\tau_{1m}}\right) \\ & + \frac{i}{2\omega_1}(\eta_2+\bar{\eta}_2)\sum_{m=1}^{M} f_{2m}\left(z_m e^{i\tau_{2m}} + \bar{z}_m e^{-i\tau_{2m}}\right) \end{aligned} \tag{10.12}$$

$$
\begin{aligned}
F_{22} + \mathcal{L}(h_{22}) - i\omega_2 h_{22} = & -\frac{\partial h_{21}}{\partial \eta_1} F_{11} - \frac{\partial h_{21}}{\partial \bar{\eta}_1} \bar{F}_{11} - \frac{\partial h_{21}}{\partial \eta_2} F_{21} - \frac{\partial h_{21}}{\partial \bar{\eta}_2} \bar{F}_{21} \\
& + \frac{i}{2\omega_2} \Big[2\delta_2(\eta_1 + \bar{\eta}_1)(h_{11} + \bar{h}_{11}) + 2\delta_3(\eta_1 + \bar{\eta}_1)(h_{21} + \bar{h}_{21}) \\
& + 2\delta_3(\eta_2 + \bar{\eta}_2)(h_{11} + \bar{h}_{11}) + 2\delta_4(\eta_2 + \bar{\eta}_2)(h_{21} + \bar{h}_{21}) \Big] \\
& + \frac{i}{2\omega_2} \left[\alpha_2(\eta_1 + \bar{\eta}_1)^3 + \alpha_3(\eta_1 + \bar{\eta}_1)^2(\eta_2 + \bar{\eta}_2) \right. \\
& \left. + 3\alpha_4(\eta_1 + \bar{\eta}_1)(\eta_2 + \bar{\eta}_2)^2 + \alpha_5(\eta_2 + \bar{\eta}_2)^3 \right] \\
& - \mu_2(\eta_2 - \bar{\eta}_2) + \frac{i}{2\omega_2}(\eta_1 + \bar{\eta}_1) \sum_{m=1}^{M} g_{1m} \left(z_m e^{i\nu_{1m}} + \bar{z}_m e^{-i\nu_{1m}} \right) \\
& + \frac{i}{2\omega_2}(\eta_2 + \bar{\eta}_2) \sum_{m=1}^{M} g_{2m} \left(z_m e^{i\nu_{2m}} + \bar{z}_m e^{-i\nu_{2m}} \right)
\end{aligned}
\tag{10.13}
$$

where

$$
\begin{aligned}
\mathcal{L}(h) = & \, i\omega_1 \left(\frac{\partial h}{\partial \eta_1} \eta_1 - \frac{\partial h}{\partial \bar{\eta}_1} \bar{\eta}_1 \right) + i\omega_2 \left(\frac{\partial h}{\partial \eta_2} \eta_2 - \frac{\partial h}{\partial \bar{\eta}_2} \bar{\eta}_2 \right) \\
& + i \sum_{m=1}^{M} \Omega_m \left(\frac{\partial h}{\partial z_m} z_m - \frac{\partial h}{\partial \bar{z}_m} \bar{z}_m \right)
\end{aligned}
$$

Next, we choose h_{11} and h_{21} to eliminate the nonresonance terms and F_{11} and F_{21} to eliminate the resonance and near-resonance terms. We start by assuming that there are no resonance and near-resonance terms and hence set F_{11} and F_{21} equal to zero. Then, we determine the conditions under which this assumption is violated. The forms of the terms in (10.10) and (10.11) suggest seeking h_{11} and h_{21} in the forms

$$
\begin{aligned}
h_{11} = & \, \Gamma_1 \eta_1^2 + \Gamma_2 \eta_1 \bar{\eta}_1 + \Gamma_3 \bar{\eta}_1^2 + \Gamma_4 \eta_2^2 + \Gamma_5 \eta_2 \bar{\eta}_2 + \Gamma_6 \bar{\eta}_2^2 + \Gamma_7 \eta_1 \eta_2 \\
& + \Gamma_8 \eta_1 \bar{\eta}_2 + \Gamma_9 \bar{\eta}_1 \eta_2 + \Gamma_{10} \bar{\eta}_1 \bar{\eta}_2
\end{aligned}
\tag{10.14}
$$

$$
\begin{aligned}
h_{21} = & \, \Lambda_1 \eta_1^2 + \Lambda_2 \eta_1 \bar{\eta}_1 + \Lambda_3 \bar{\eta}_1^2 + \Lambda_4 \eta_2^2 + \Lambda_5 \eta_2 \bar{\eta}_2 + \Lambda_6 \bar{\eta}_2^2 + \Lambda_7 \eta_1 \eta_2 \\
& + \Lambda_8 \eta_1 \bar{\eta}_2 + \Lambda_9 \bar{\eta}_1 \eta_2 + \Lambda_{10} \bar{\eta}_1 \bar{\eta}_2
\end{aligned}
\tag{10.15}
$$

Substituting (10.14) and (10.5) into (10.10) and (10.11), putting F_{11} and F_{21} equal to zero, and choosing the Γ_m and Λ_m to eliminate all of the terms, we obtain

$$
\begin{aligned}
& \Gamma_1 = \frac{\delta_1}{2\omega_1^2}, \quad \Gamma_2 = -\frac{\delta_1}{\omega_1^2}, \quad \Gamma_3 = -\frac{\delta_1}{6\omega_1^2}, \quad \Gamma_4 = \frac{\delta_3}{2\omega_1(2\omega_2 - \omega_1)}, \\
& \Gamma_5 = -\frac{\delta_3}{\omega_1^2}, \quad \Gamma_6 = -\frac{\delta_3}{2\omega_1(2\omega_2 + \omega_1)}, \quad \Gamma_7 = \frac{\delta_2}{\omega_1 \omega_2}, \quad \Gamma_8 = -\frac{\delta_2}{\omega_1 \omega_2}, \\
& \Gamma_9 = \frac{\delta_2}{\omega_1(\omega_2 - 2\omega_1)}, \quad \Gamma_{10} = -\frac{\delta_2}{\omega_1(\omega_2 + 2\omega_1)}
\end{aligned}
\tag{10.16}
$$

$$\Lambda_1 = \frac{\delta_2}{2\omega_2(2\omega_1 - \omega_2)}\,, \quad \Lambda_2 = -\frac{\delta_2}{\omega_2^2}\,, \quad \Lambda_3 = -\frac{\delta_2}{2\omega_2(\omega_2 + 2\omega_1)}\,,$$
$$\Lambda_4 = \frac{\delta_4}{2\omega_2^2}\,, \quad \Lambda_5 = -\frac{\delta_4}{\omega_2^2}\,, \quad \Lambda_6 = -\frac{\delta_4}{6\omega_2^2}\,, \quad \Lambda_7 = \frac{\delta_3}{\omega_1\omega_2}\,,$$
$$\Lambda_8 = \frac{\delta_3}{\omega_2(\omega_1 - 2\omega_2)}\,, \quad \Lambda_9 = -\frac{\delta_3}{\omega_1\omega_2}\,, \quad \Lambda_{10} = -\frac{\delta_3}{\omega_2(2\omega_2 + \omega_1)} \tag{10.17}$$

It follows from (10.16) and (10.17) that the transformation (10.8) breaks down when $\omega_2 \approx 2\omega_1$ or $\omega_1 \approx 2\omega_2$ and hence (10.10) and (10.11) contain near-resonance terms in these cases. These cases correspond to two-to-one internal resonances, which need to be treated separately. This is done in Section 10.5.

It follows from (10.14)–(10.17) that

$$\begin{aligned} h_{11} + \bar{h}_{11} = {} & \frac{\delta_1}{3\omega_1^2}\left(\eta_1^2 + \bar{\eta}_1^2\right) - \frac{2\delta_1}{\omega_1^2}\eta_1\bar{\eta}_1 + \frac{\delta_3}{4\omega_2^2 - \omega_1^2}\left(\eta_2^2 + \bar{\eta}_2^2\right) \\ & - \frac{2\delta_3}{\omega_1^2}\eta_2\bar{\eta}_2 + \frac{2\delta_2}{\omega_2(\omega_2 + 2\omega_1)}\left(\eta_1\eta_2 + \bar{\eta}_1\bar{\eta}_2\right) \\ & + \frac{2\delta_2}{\omega_2(\omega_2 - 2\omega_1)}\left(\eta_2\bar{\eta}_1 + \eta_1\bar{\eta}_2\right) \end{aligned} \tag{10.18}$$

$$\begin{aligned} h_{21} + \bar{h}_{21} = {} & \frac{\delta_2}{4\omega_1^2 - \omega_2^2}\left(\eta_1^2 + \bar{\eta}_1^2\right) - \frac{2\delta_2}{\omega_2^2}\eta_1\bar{\eta}_1 + \frac{\delta_4}{3\omega_2^2}\left(\eta_2^2 + \bar{\eta}_2^2\right) \\ & - \frac{2\delta_4}{\omega_2^2}\eta_2\bar{\eta}_2 + \frac{2\delta_3}{\omega_1(\omega_1 + 2\omega_2)}\left(\eta_2\eta_1 + \bar{\eta}_1\bar{\eta}_2\right) \\ & + \frac{2\delta_3}{\omega_1(\omega_1 - 2\omega_2)}\left(\eta_2\bar{\eta}_1 + \eta_1\bar{\eta}_2\right) \end{aligned} \tag{10.19}$$

Next, we substitute (10.18) and (10.19) into (10.12) and (10.13) and choose h_{12} and h_{22} to eliminate all of the nonresonance terms, thereby leaving F_{12} and F_{22} with all of the resonance and near-resonance terms. If we stop at $O(\epsilon^2)$, then we do not need to determine h_{12} and h_{22}, and hence all that we need to do is to determine F_{12} and F_{22}. Inspecting the right-hand sides of (10.12) and (10.13), we find that the following resonances are possible:

- $\omega_2 \approx \omega_1$: One-to-one internal resonance,
- $\omega_2 \approx 3\omega_1$: Three-to-one internal resonance,
- $\Omega_n \approx 2\omega_n$: Principal parametric resonance,
- $\Omega_m \approx \omega_1 \pm \omega_1$: Combination parametric resonance.

In what follows, we consider the different internal resonance cases in conjunction with the parametric resonances:

$$\Omega_m \approx \omega_2 - \omega_1\,, \quad \Omega_n \approx \omega_2 + \omega_1\,, \quad \Omega_p \approx 2\omega_1\,, \quad \text{and} \quad \Omega_q \approx 2\omega_2$$

10.2 The Case of No Internal Resonance

Choosing F_{12} and F_{22} to eliminate all of the resonance and near-resonance terms in (10.12) and (10.13), we obtain

$$F_{12} = -\mu_1\eta_1 + \frac{4i}{\omega_1}\left(S_{11}\eta_1^2\bar{\eta}_1 + S_{12}\eta_2\bar{\eta}_2\eta_1\right) + \frac{i}{2\omega_1}\left(f_{1p}\bar{\eta}_1 z_p e^{i\tau_{1p}} + f_{2m}\eta_2\bar{z}_m e^{-i\tau_{2m}} + f_{2n}\bar{\eta}_2 z_n e^{i\tau_{2n}}\right) \tag{10.20}$$

$$F_{22} = -\mu_2\eta_2 + \frac{4i}{\omega_2}\left(S_{12}\eta_1\bar{\eta}_1\eta_2 + S_{22}\eta_2^2\bar{\eta}_2\right) + \frac{i}{2\omega_2}\left(g_{1m}\eta_1 z_m e^{i\nu_{1m}} + g_{1n}\bar{\eta}_1 z_n e^{i\nu_{1n}} + g_{2q}\bar{\eta}_2 z_q e^{i\nu_{2q}}\right) \tag{10.21}$$

where

$$8S_{11} = 3\alpha_1 - \frac{10\delta_1^2}{3\omega_1^2} - \frac{4\delta_2^2}{\omega_2^2} + \frac{2\delta_2^2}{4\omega_1^2 - \omega_2^2} \tag{10.22}$$

$$8S_{12} = 2\alpha_3 - \frac{4\delta_1\delta_3}{\omega_1^2} - \frac{4\delta_2\delta_4}{\omega_2^2} + \frac{8\delta_2^2}{\omega_2^2 - 4\omega_1^2} + \frac{8\delta_3^2}{\omega_1^2 - 4\omega_2^2} \tag{10.23}$$

$$8S_{22} = 3\alpha_5 - \frac{10\delta_4^2}{3\omega_2^2} - \frac{4\delta_3^2}{\omega_1^2} + \frac{2\delta_3^2}{4\omega_2^2 - \omega_1^2} \tag{10.24}$$

Substituting (10.8), (10.18), and (10.19) into (10.4) yields

$$u_1 = \eta_1 + \bar{\eta}_1 + \epsilon\left[\frac{\delta_1}{3\omega_1^2}\left(\eta_1^2 + \bar{\eta}_1^2\right) - \frac{2\delta_1}{\omega_1^2}\eta_1\bar{\eta}_1 + \frac{\delta_3}{4\omega_2^2 - \omega_1^2}\left(\eta_2^2 + \bar{\eta}_2^2\right) - \frac{2\delta_3}{\omega_1^2}\eta_2\bar{\eta}_2 + \frac{2\delta_2}{\omega_2(\omega_2 + 2\omega_1)}\left(\eta_1\eta_2 + \bar{\eta}_1\bar{\eta}_2\right) + \frac{2\delta_2}{\omega_2(\omega_2 - 2\omega_1)}\left(\eta_2\bar{\eta}_1 + \eta_1\bar{\eta}_2\right)\right] + \cdots \tag{10.25}$$

$$u_2 = \eta_2 + \bar{\eta}_2 + \epsilon\left[\frac{\delta_2}{4\omega_1^2 - \omega_2^2}\left(\eta_1^2 + \bar{\eta}_1^2\right) - \frac{2\delta_2}{\omega_2^2}\eta_1\bar{\eta}_1 + \frac{\delta_4}{3\omega_2^2}\left(\eta_2^2 + \bar{\eta}_2^2\right) - \frac{2\delta_4}{\omega_2^2}\eta_2\bar{\eta}_2 + \frac{2\delta_3}{\omega_1(\omega_1 + 2\omega_2)}\left(\eta_2\eta_1 + \bar{\eta}_2\bar{\eta}_1\right) + \frac{2\delta_3}{\omega_1(\omega_1 - 2\omega_2)}\left(\eta_2\bar{\eta}_1 + \eta_1\bar{\eta}_2\right)\right] + \cdots \tag{10.26}$$

Substituting for the F_{mn} into (10.9), we obtain the normal form

$$\dot{\eta}_1 = i\omega_1\eta_1 - \epsilon^2\mu_1\eta_1 + \frac{4i\epsilon^2}{\omega_1}\left(S_{11}\eta_1^2\bar{\eta}_1 + S_{12}\eta_2\bar{\eta}_2\eta_1\right) + \frac{i\epsilon^2}{2\omega_1}\left(f_{1p}\bar{\eta}_1 z_p e^{i\tau_{1p}} + f_{2m}\eta_2\bar{z}_m e^{-i\tau_{2m}} + f_{2n}\bar{\eta}_2 z_n e^{i\tau_{2n}}\right) + \cdots \tag{10.27}$$

$$\dot{\eta}_2 = i\omega_2\eta_2 - \epsilon^2\mu_2\eta_2 + \frac{4i\epsilon^2}{\omega_2}\left(S_{12}\eta_1\bar{\eta}_1\eta_2 + S_{22}\eta_2^2\bar{\eta}_2\right) + \frac{i\epsilon^2}{2\omega_2}\left(g_{1m}\eta_1 z_m e^{i\nu_{1m}} + g_{1n}\bar{\eta}_1 z_n e^{i\nu_{1n}} + g_{2q}\bar{\eta}_2 z_q e^{i\nu_{2q}}\right) + \cdots \quad (10.28)$$

Replacing the η_n in (10.25)–(10.28) with $A_n e^{i\omega_n t}$, we obtain the same results found by using the method of multiple scales (Nayfeh and Jebril, 1987). Moreover, in the absence of damping and forcing, (10.27) and (10.28) are derivable from the Lagrangian

$$L = i\omega_1\left(\eta_1\dot{\bar{\eta}}_1 - \dot{\eta}_1\bar{\eta}_1\right) + i\omega_2\left(\eta_2\dot{\bar{\eta}}_2 - \dot{\eta}_2\bar{\eta}_2\right) - 2\omega_1^2\eta_1\bar{\eta}_1 - 2\omega_2^2\eta_2\bar{\eta}_2 - 4\epsilon^2\left(S_{11}\eta_1^2\bar{\eta}_1^2 + 2S_{12}\eta_1\bar{\eta}_1\eta_2\bar{\eta}_2 + S_{22}\eta_2^2\bar{\eta}_2^2\right)$$

10.3 The Case of Three-to-One Internal Resonance

The three-to-one internal resonance produces the additional near-resonance terms

$$\frac{4i}{\omega_1}S_{13}\eta_2\bar{\eta}_1^2 \quad \text{and} \quad \frac{4i}{\omega_2}S_{23}\eta_1^3$$

in (10.12) and (10.13), respectively, and hence F_{12} and F_{22} are chosen to contain these additional terms, respectively, where

$$8S_{13} = 3\alpha_2 + \frac{2\delta_1\delta_2}{3\omega_1^2} + \frac{4\delta_1\delta_2}{\omega_2(\omega_2 - 2\omega_1)} + \frac{4\delta_2\delta_3}{\omega_1(\omega_1 - 2\omega_2)} + \frac{2\delta_2\delta_3}{4\omega_1^2 - \omega_2^2} \quad (10.29)$$

$$8S_{23} = \alpha_2 + \frac{2\delta_1\delta_2}{3\omega_1^2} + \frac{2\delta_2\delta_3}{4\omega_1^2 - \omega_2^2} \quad (10.30)$$

Consequently, to the second approximation, u_1 and u_2 are given by (10.25) and (10.26), where η_1 and η_2 are given by the normal form

$$\dot{\eta}_1 = i\omega_1\eta_1 - \epsilon^2\mu_1\eta_1 + \frac{4i\epsilon^2}{\omega_1}\left(S_{11}\eta_1^2\bar{\eta}_1 + S_{12}\eta_2\bar{\eta}_2\eta_1 + S_{13}\eta_2\bar{\eta}_1^2\right) + \frac{i\epsilon^2}{2\omega_1}\left(f_{1p}\bar{\eta}_1 z_p e^{i\tau_{1p}} + f_{2m}\eta_2\bar{z}_m e^{-i\tau_{2m}} + f_{2n}\bar{\eta}_2 z_n e^{i\tau_{2n}}\right) + \cdots \quad (10.31)$$

$$\dot{\eta}_2 = i\omega_2\eta_2 - \epsilon^2\mu_2\eta_2 + \frac{4i\epsilon^2}{\omega_2}\left(S_{12}\eta_1\bar{\eta}_1\eta_2 + S_{22}\eta_2^2\bar{\eta}_2 + S_{23}\eta_1^3\right) + \frac{i\epsilon^2}{2\omega_2}\left(g_{1m}\eta_1 z_m e^{i\nu_{1m}} + g_{1n}\bar{\eta}_1\bar{z}_n e^{i\nu_{1n}} + g_{2q}\bar{\eta}_2 z_q e^{i\nu_{2q}}\right) + \cdots \quad (10.32)$$

As $\omega_2 \to 3\omega_1$, (10.29) and (10.30) reduce to

$$8S_{13} = 3\alpha_2 + \frac{2\delta_1\delta_2}{\omega_1^2} - \frac{6\delta_2\delta_3}{5\omega_1^2}$$

$$8S_{23} = \alpha_2 + \frac{2\delta_1\delta_2}{3\omega_1^2} - \frac{2\delta_2\delta_3}{5\omega_1^2}$$

and thus $S_{13} = 3S_{23}$. Therefore, in the absence of damping and forcing, (10.31) and (10.32) are derivable from the Lagrangian

$$L = i\omega_1(\eta_1\dot{\bar{\eta}}_1 - \dot{\eta}_1\bar{\eta}_1) + i\omega_2(\eta_2\dot{\bar{\eta}}_2 - \dot{\eta}_2\bar{\eta}_2) - 2\omega_1^2\eta_1\bar{\eta}_1 - 2\omega_2^2\eta_2\bar{\eta}_2 \\ - 4\epsilon^2\left[S_{11}\eta_1^2\bar{\eta}_1^2 + 2S_{12}\eta_1\bar{\eta}_1\eta_2\bar{\eta}_2 + S_{22}\eta_2^2\bar{\eta}_2^2 + S_{23}\left(\eta_2\bar{\eta}_1^3 + \bar{\eta}_2\eta_1^3\right)\right]$$

10.4
The Case of One-to-One Internal Resonance

Choosing F_{12} and F_{22} to eliminate the resonance near-resonance terms in (10.12) and (10.13) and then substituting the results into (10.9), we obtain the following normal form:

$$\dot{\eta}_1 = i\omega_1\eta_1 - \epsilon^2\mu_1\eta_1 + \frac{4i\epsilon^2}{\omega_1}\left(S_{11}\eta_1^2\bar{\eta}_1 + S_{12}\eta_2\bar{\eta}_2\eta_1 + S_{13}\eta_1^2\bar{\eta}_2\right. \\ \left. + S_{14}\eta_2^2\bar{\eta}_1 + S_{15}\eta_1\bar{\eta}_1\eta_2 + S_{16}\eta_2^2\bar{\eta}_2\right) \\ + \frac{i\epsilon^2}{2\omega_1}\left(f_{1p}\bar{\eta}_1 z_p e^{i\tau_{1p}} + f_{2m}\eta_2\bar{z}_m e^{-i\tau_{2m}} + f_{2n}\bar{\eta}_2 z_n e^{i\tau_{2n}}\right) \tag{10.33}$$

$$\dot{\eta}_2 = i\omega_2\eta_2 - \epsilon^2\mu_2\eta_2 + \frac{4i\epsilon^2}{\omega_2}\left(S_{12}\eta_2\eta_1\bar{\eta}_1 + S_{22}\eta_2^2\bar{\eta}_2 + S_{23}\eta_1^2\bar{\eta}_1\right. \\ \left. + S_{24}\eta_2^2\bar{\eta}_1 + S_{25}\eta_1^2\bar{\eta}_2 + S_{26}\eta_2\bar{\eta}_2\eta_1\right) \\ + \frac{i\epsilon^2}{2\omega_2}\left(g_{1m}\eta_1 z_m e^{i\nu_{1m}} + g_{1n}\bar{\eta}_1 z_n e^{i\nu_{1n}} + g_{2q}\bar{\eta}_2 z_q e^{i\nu_{2q}}\right) \tag{10.34}$$

where

$$S_{13} = 3\alpha_2 + \frac{2\delta_1\delta_2}{3\omega_1^2} + \frac{4\delta_2\delta_3}{\omega_1(\omega_1 - 2\omega_2)} + \frac{4\delta_1\delta_2}{\omega_2(\omega_2 - 2\omega_1)} + \frac{2\delta_2\delta_3}{4\omega_1^2 - \omega_2^2} \tag{10.35}$$

$$S_{14} = \alpha_3 + \frac{2\delta_1\delta_3}{4\omega_2^2 - \omega_1^2} + \frac{2\delta_2\delta_4}{3\omega_2^2} + \frac{4\delta_2^2}{\omega_2(\omega_2 - 2\omega_1)} + \frac{4\delta_3^2}{\omega_1(\omega_1 - 2\omega_2)} \tag{10.36}$$

$$S_{15} = 6\alpha_2 + \frac{8\delta_1\delta_2}{\omega_2^2 - 4\omega_1^2} + \frac{8\delta_2\delta_3}{\omega_1^2 - 4\omega_2^2} - \frac{4\delta_1\delta_2}{\omega_1^2} - \frac{4\delta_2\delta_3}{\omega_2^2} \tag{10.37}$$

$$S_{16} = 3\alpha_4 - \frac{10\delta_3\delta_4}{3\omega_2^2} - \frac{4\delta_2\delta_3}{\omega_1^2} + \frac{2\delta_2\delta_3}{4\omega_2^2 - \omega_1^2} \tag{10.38}$$

$$S_{23} = 3\alpha_2 - \frac{10\delta_1\delta_2}{3\omega_1^2} - \frac{4\delta_2\delta_3}{\omega_2^2} + \frac{2\delta_2\delta_3}{4\omega_1^2 - \omega_2^2} \tag{10.39}$$

$$S_{24} = 3\alpha_4 + \frac{2\delta_2\delta_3}{4\omega_2^2 - \omega_1^2} + \frac{2\delta_3\delta_4}{3\omega_2^2} + \frac{4\delta_2\delta_3}{\omega_2(\omega_2 - 2\omega_1)} + \frac{4\delta_3\delta_4}{\omega_1(\omega_1 - 2\omega_2)} \quad (10.40)$$

$$S_{25} = \alpha_3 + \frac{2\delta_1\delta_3}{3\omega_1^2} + \frac{2\delta_2\delta_4}{4\omega_1^2 - \omega_2^2} + \frac{4\delta_2^2}{\omega_2(\omega_2 - 2\omega_1)} + \frac{4\delta_3^2}{\omega_1(\omega_1 - 2\omega_2)} \quad (10.41)$$

$$S_{26} = 6\alpha_4 - \frac{4\delta_1\delta_3}{\omega_1^2} - \frac{4\delta_2\delta_4}{\omega_2^2} + \frac{8\delta_2^2}{\omega_2^2 - 4\omega_1^2} + \frac{8\delta_3^2}{\omega_1^2 - 4\omega_2^2} \quad (10.42)$$

Therefore, to the second approximation, u_1 and u_2 are given by (10.25) and (10.26), where η_1 and η_2 are given by (10.33) and (10.34).

As $\omega_2 \to \omega_1$, (10.35)–(10.42) reduce to

$$8S_{13} = 3\alpha_2 - \frac{10\delta_1\delta_2}{3\omega_1^2} - \frac{10\delta_2\delta_3}{3\omega_1^2}$$

$$8S_{14} = \alpha_3 - \frac{4\delta_2^2}{\omega_1^2} + \frac{2\delta_1\delta_3}{3\omega_1^2} - \frac{4\delta_3^2}{\omega_1^2} + \frac{2\delta_2\delta_4}{3\omega_1^2}$$

$$8S_{15} = 6\alpha_2 - \frac{20\delta_1\delta_2}{3\omega_1^2} - \frac{20\delta_2\delta_3}{3\omega_1^2}$$

$$8S_{16} = 3\alpha_4 - \frac{10\delta_2\delta_3}{3\omega_1^2} - \frac{10\delta_3\delta_4}{3\omega_1^2}$$

$$8S_{23} = 3\alpha_2 - \frac{10\delta_1\delta_2}{3\omega_1^2} - \frac{10\delta_2\delta_3}{3\omega_1^2}$$

$$8S_{24} = 3\alpha_4 - \frac{10\delta_2\delta_3}{3\omega_1^2} - \frac{10\delta_3\delta_4}{3\omega_1^2}$$

$$8S_{25} = \alpha_3 - \frac{4\delta_2^2}{\omega_1^2} + \frac{2\delta_1\delta_3}{3\omega_1^2} - \frac{4\delta_3^2}{\omega_1^2} + \frac{2\delta_2\delta_4}{3\omega_1^2}$$

$$8S_{26} = 6\alpha_4 - \frac{20\delta_2\delta_3}{3\omega_1^2} - \frac{20\delta_3\delta_4}{3\omega_1^2}$$

Thus,

$$S_{13} = S_{23} = \tfrac{1}{2}S_{15}\,, \quad S_{16} = S_{24} = \tfrac{1}{2}S_{26}\,, \quad \text{and} \quad S_{14} = S_{25}$$

Therefore, in the absence of damping and forcing, (10.33) and (10.34) are derivable from the Lagrangian

$$\begin{aligned} L = {} & i\omega_1(\eta_1\dot{\bar{\eta}}_1 - \dot{\eta}_1\bar{\eta}_1) + i\omega_2(\eta_2\dot{\bar{\eta}}_2 - \dot{\eta}_2\bar{\eta}_2) - 2\omega_1^2\eta_1\bar{\eta}_1 - 2\omega_2^2\eta_2\bar{\eta}_2 \\ & - 4\epsilon^2\left[S_{11}\eta_1^2\bar{\eta}_1^2 + S_{22}\eta_2^2\bar{\eta}_2^2 + 2S_{12}\eta_1\bar{\eta}_1\eta_2\bar{\eta}_2\right. \\ & + 2S_{13}\left(\eta_1^2\bar{\eta}_1\bar{\eta}_2 + \bar{\eta}_1^2\eta_1\eta_2\right) \\ & \left. + S_{14}\left(\eta_2^2\bar{\eta}_1^2 + \eta_1^2\bar{\eta}_2^2\right) + 2S_{16}\left(\eta_2^2\bar{\eta}_2\bar{\eta}_1 + \bar{\eta}_2^2\eta_2\eta_1\right)\right] \end{aligned}$$

10.5
The Case of Two-to-One Internal Resonance

We consider the case of $\omega_2 \approx 2\omega_1$. Specifically, we let $\omega_2 = 2\omega_1 + \epsilon\sigma$, where σ is a detuning parameter. The case of $\omega_1 \approx 2\omega_2$ can be treated in a similar fashion. It follows from (10.16) and (10.17) that Γ_9 and Λ_1 have small divisors and hence the term $\eta_2\bar{\eta}_1$ in (10.10) and the term η_1^2 in (10.11) are near-resonance terms. For a uniform expansion, we choose F_{11} and F_{21} to eliminate these terms; that is,

$$F_{11} = \frac{i\delta_2}{\omega_1}\eta_2\bar{\eta}_1 \quad \text{and} \quad F_{21} = \frac{i\delta_2}{2\omega_2}\eta_1^2 \tag{10.43}$$

Then, substituting (10.14) and (10.15) into (10.10) and (10.11) and eliminating the nonresonance terms, we find that the Γ_m and Λ_n, except Γ_9 and Λ_1, are given by (10.16) and (10.17).

Substituting (10.14)–(10.17) into (10.4), we obtain

$$\begin{aligned} u_1 = \eta_1 + \bar{\eta}_1 + \epsilon\bigg[&\frac{\delta_1}{3\omega_1^2}\left(\eta_1^2 + \bar{\eta}_1^2\right) - \frac{2\delta_1}{\omega_1^2}\eta_1\bar{\eta}_1 + \frac{\delta_3}{4\omega_2^2 - \omega_1^2}\left(\eta_2^2 + \bar{\eta}_2^2\right) \\ &- \frac{2\delta_3}{\omega_1^2}\eta_2\bar{\eta}_2 + \frac{2\delta_2}{\omega_2(\omega_2 + 2\omega_1)}\left(\eta_1\eta_2 + \bar{\eta}_1\bar{\eta}_2\right) \\ &- \frac{\delta_2}{\omega_1\omega_2}\left(\eta_1\bar{\eta}_2 + \bar{\eta}_1\eta_2\right)\bigg] + \cdots \end{aligned} \tag{10.44}$$

$$\begin{aligned} u_2 = \eta_2 + \bar{\eta}_2 - \epsilon\bigg[&\frac{\delta_2}{2\omega_2(\omega_2 + 2\omega_1)}\left(\eta_1^2 + \bar{\eta}_1^2\right) + \frac{2\delta_2}{\omega_2^2}\eta_1\bar{\eta}_1 \\ &- \frac{\delta_4}{3\omega_2^2}\left(\eta_2^2 + \bar{\eta}_2^2\right) + \frac{2\delta_4}{\omega_2^2}\eta_2\bar{\eta}_2 - \frac{2\delta_3}{\omega_1(\omega_1 + 2\omega_2)}\left(\eta_2\eta_1 - \bar{\eta}_2\bar{\eta}_1\right) \\ &- \frac{2\delta_3}{\omega_2(\omega_1 - 2\omega_2)}\left(\eta_2\bar{\eta}_1 + \eta_1\bar{\eta}_2\right)\bigg] + \cdots \end{aligned} \tag{10.45}$$

Substituting (10.14)–(10.17) into (10.12) and (10.13) and choosing F_{12} and F_{22} to eliminate the resonance and near-resonance terms, we obtain

$$\begin{aligned} F_{12} = &-\mu_1\eta_1 + \frac{4i}{\omega_1}\left(S_{11}\eta_1^2\bar{\eta}_1 + S_{12}\eta_2\bar{\eta}_2\eta_1\right) \\ &+ \frac{i}{2\omega_1}\left[f_{1p}\bar{\eta}_1 z_p e^{i\tau_{1p}} + f_{2m}\eta_2\bar{z}_m e^{-i\tau_{2m}} + f_{2n}\bar{\eta}_2 z_n e^{i\tau_{2n}}\right] \end{aligned} \tag{10.46}$$

$$\begin{aligned} F_{22} = &-\mu_2\eta_2 + \frac{4i}{\omega_2}\left(S_{21}\eta_1\bar{\eta}_1\eta_2 + S_{22}\eta_2^2\bar{\eta}_2\right) \\ &+ \frac{i}{2\omega_2}\left[g_{1m}\eta_1 z_m e^{i\nu_{1m}} + g_{1n}\bar{\eta}_2\bar{z}_n e^{-i\nu_{1n}} + g_{2q}\bar{\eta}_2 z_q e^{i\nu_{2q}}\right] \end{aligned} \tag{10.47}$$

where

$$8S_{11} = 3\alpha_1 - \frac{10\delta_1^2}{3\omega_1^2} - \frac{4\delta_2^2}{\omega_2^2} - \frac{\delta_2^2}{\omega_2(\omega_2 + 2\omega_1)} \tag{10.48}$$

$$8S_{12} = 8S_{21} = 2\alpha_3 - \frac{4\delta_1\delta_3}{\omega_1^2} - \frac{4\delta_2\delta_4}{\omega_2^2} + \frac{8\delta_3^2}{\omega_1^2 - 4\omega_2^2} - \frac{2\delta_2^2}{\omega_1(\omega_2 + 2\omega_1)} \tag{10.49}$$

$$8S_{22} = 3\alpha_5 - \frac{10\delta_4^2}{3\omega_2^2} - \frac{4\delta_3^2}{\omega_1^2} + \frac{2\delta_3^2}{4\omega_2^2 - \omega_1^2} \tag{10.50}$$

Therefore, to the second approximation, u_1 and u_2 are given by (10.44) and (10.45) where η_1 and η_2 are given by the normal form

$$\dot{\eta}_1 = i\omega_1\eta_1 + \frac{i\epsilon\delta_2}{\omega_1}\eta_2\bar{\eta}_1 - \epsilon^2\mu_1\eta_1 + \frac{4i\epsilon^2}{\omega_1}\left(S_{11}\eta_1^2\bar{\eta}_1 + S_{12}\eta_2\bar{\eta}_2\eta_1\right)$$
$$+ \frac{i\epsilon^2}{2\omega_1}\left[f_{1p}\bar{\eta}_1 z_p e^{i\tau_{1p}} + f_{2m}\eta_2\bar{z}_m e^{-i\tau_{2m}} + f_{2n}\bar{\eta}_2 z_n e^{i\tau_{2n}}\right] \tag{10.51}$$

$$\dot{\eta}_2 = i\omega_2\eta_2 + \frac{i\epsilon\delta_2}{2\omega_2}\eta_1^2 - \epsilon^2\mu_2\eta_2 + \frac{4i\epsilon^2}{\omega_2}\left(S_{21}\eta_1\bar{\eta}_1\eta_2 + S_{22}\eta_2^2\bar{\eta}_2\right)$$
$$+ \frac{i\epsilon^2}{2\omega_2}\left[g_{1m}\eta_1 z_m e^{i\nu_{1m}} + g_{1n}\bar{\eta}_2 z_n e^{i\nu_{1n}} + g_{2q}\bar{\eta}_2 z_q e^{i\nu_{2q}}\right] \tag{10.52}$$

In the absence of damping and forcing, (10.51) and (10.52) are derivable from the Lagrangian

$$L = i\omega_1\left(\eta_1\dot{\bar{\eta}}_1 - \dot{\eta}_1\bar{\eta}_1\right) + i\omega_2\left(\eta_2\dot{\bar{\eta}}_2 - \dot{\eta}_2\bar{\eta}_2\right) - 2\omega_1^2\eta_1\bar{\eta}_1 - 2\omega_2^2\eta_2\bar{\eta}_2$$
$$- \epsilon\delta_2\left(\eta_2\bar{\eta}_1^2 + \bar{\eta}_2\eta_1^2\right) - 4\epsilon^2\left(S_{11}\eta_1^2\bar{\eta}_1^2 + 2S_{12}\eta_1\bar{\eta}_1\eta_2\bar{\eta}_2 + S_{22}\eta_2^2\bar{\eta}_2^2\right)$$

10.6
Method of Multiple Scales

To describe the method without algebraic complication, we consider the following special case of (10.1) and (10.2):

$$\ddot{u}_1 + \omega_1^2 u_1 = -2\epsilon\delta_2 u_1 u_2 \tag{10.53}$$

$$\ddot{u}_2 + \omega_2^2 u_2 = -\epsilon\delta_2 u_1^2 \tag{10.54}$$

where $\omega_2 \approx 2\omega_1$. We describe three implementations of the method of multiple scales: (a) treatment of the governing equations in second-order form, (b) treatment of the governing equations in state-space form, and (a) treatment of the governing equations in complex-valued form.

10.6.1
Second-Order Form

We seek a second-order approximation of the solution of (10.53) and (10.54) in the form

$$u_n(t;\epsilon) = \sum_{m=0}^{2} \epsilon^m u_{nm}(T_0, T_1, T_2) + \cdots \tag{10.55}$$

Substituting (10.55) into (10.53) and (10.54) and equating coefficients of like powers of ϵ, we obtain:

Order (ϵ^0)

$$D_0^2 u_{10} + \omega_1^2 u_{10} = 0 \tag{10.56}$$

$$D_0^2 u_{20} + \omega_2^2 u_{20} = 0 \tag{10.57}$$

Order (ϵ)

$$D_0^2 u_{11} + \omega_1^2 u_{11} = -2D_0 D_1 u_{10} - 2\delta_2 u_{10} u_{20} \tag{10.58}$$

$$D_0^2 u_{21} + \omega_2^2 u_{21} = -2D_0 D_1 u_{20} - \delta_2 u_{10}^2 \tag{10.59}$$

Order (ϵ^2)

$$\begin{aligned} D_0^2 u_{12} + \omega_1^2 u_{12} = &-2D_0 D_1 u_{11} - D_1^2 u_{10} - 2\delta_2 u_{11} u_{20} \\ &- 2D_0 D_2 u_{10} - 2\delta_2 u_{10} u_{21} \end{aligned} \tag{10.60}$$

$$D_0^2 u_{22} + \omega_2^2 u_{22} = -2D_0 D_1 u_{21} - D_1^2 u_{20} - 2D_0 D_2 u_{20} - 2\delta_2 u_{10} u_{11} \tag{10.61}$$

The general solutions of (10.56) and (10.57) can be expressed as

$$u_{m0} = A_m(T_1, T_2)\, e^{i\omega_m T_0} + \text{cc} \tag{10.62}$$

Then, (10.58) and (10.59) become

$$\begin{aligned} D_0^2 u_{11} + \omega_1^2 u_{11} = &-2i\omega_1 D_1 A_1 e^{i\omega_1 T_0} - 2\delta_2 A_2 A_1 e^{i(\omega_2+\omega_1)T_0} \\ &- 2\delta_2 A_2 \bar{A}_1 e^{i(\omega_2-\omega_1)T_0} + \text{cc} \end{aligned} \tag{10.63}$$

$$\begin{aligned} D_0^2 u_{21} + \omega_2^2 u_{21} = &-2i\omega_2 D_1 A_2 e^{i\omega_2 T_0} - \delta_2 A_1^2 e^{2i\omega_1 T_0} \\ &- \delta_2 A_1 \bar{A}_1 + \text{cc} \end{aligned} \tag{10.64}$$

Any particular solution of (10.63) and (10.64) contains secular and small-divisor terms when $\omega_2 \approx 2\omega_1$. To quantify the nearness of ω_2 to $2\omega_1$, we introduce the detuning parameter σ defined by

$$\omega_2 = 2\omega_1 + \epsilon\sigma \tag{10.65}$$

Using (10.65) in eliminating the terms that lead to secular terms from (10.63) and (10.64), we have

$$i\omega_1 D_1 A_1 = -\delta_2 A_2 \bar{A}_1 e^{i\sigma T_1} \tag{10.66}$$

$$2i\omega_2 D_1 A_2 = -\delta_2 A_1^2 e^{-i\sigma T_1} \tag{10.67}$$

Then, the general solutions of (10.63) and (10.64) can be expressed as

$$u_{11} = B_1(T_1, T_2)\, e^{i\omega_1 T_0} + \frac{2\delta_2}{\omega_2(\omega_2 + 2\omega_1)} A_2 A_1 e^{i(\omega_2+\omega_1)T_0} + \text{cc} \tag{10.68}$$

$$u_{21} = B_2(T_1, T_2)\, e^{i\omega_2 T_0} - \frac{\delta_2}{\omega_2^2} A_1 \bar{A}_1 + \text{cc} \tag{10.69}$$

Substituting (10.62) and (10.66)–(10.69) into (10.60) and (10.61) and eliminating the terms that lead to secular terms, we obtain

$$\begin{aligned} &-2i\omega_1 D_2 A_1 - 2i\omega_1 D_1 B_1 - D_1^2 A_1 \\ &= 2\delta_2 \left(A_2 \bar{B}_1 + B_2 \bar{A}_1\right) e^{i\sigma T_1} \\ &\quad - \frac{4\delta_2^2}{\omega_2^2} A_1^2 \bar{A}_1 + \frac{4\delta_2^2}{\omega_2(\omega_2 + 2\omega_1)} A_1 A_2 \bar{A}_2 \end{aligned} \tag{10.70}$$

$$\begin{aligned} -2i\omega_2 D_2 A_2 - 2i\omega_2 D_1 B_2 - D_1^2 A_2 = 2\delta_2 A_1 B_1 e^{-i\sigma T_1} \\ + \frac{4\delta_2^2}{\omega_2(\omega_2 + 2\omega_1)} A_2 A_1 \bar{A}_1 \end{aligned} \tag{10.71}$$

Using (10.66) and (10.67) to eliminate $D_1^2 A_1$ and $D_1^2 A_2$ from (10.70) and (10.71) and using the method of reconstitution, we obtain the modulation equations

$$\begin{aligned} 2i\omega_1 \dot{A}_1 = -2\epsilon\delta_2 A_2 \bar{A}_1 e^{i\epsilon\sigma t} + \epsilon^2 \bigg\{ & -2i\omega_1 D_1 B_1 - 2\delta_2 \left(A_2 \bar{B}_1 + B_2 \bar{A}_1\right) e^{i\epsilon\sigma t} \\ & + \frac{\sigma\delta_2}{\omega_1} A_2 \bar{A}_1 e^{i\epsilon\sigma t} + \frac{4\delta_2^2}{\omega_2^2}\left(1 + \frac{\omega_2}{8\omega_1}\right) A_1^2 \bar{A}_1 \\ & - \frac{\delta_2^2}{\omega_1^2}\left[1 + \frac{4\omega_1^2}{\omega_2(\omega_2 + 2\omega_1)}\right] A_1 A_2 \bar{A}_2 \bigg\} + \cdots \end{aligned} \tag{10.72}$$

$$\begin{aligned} 2i\omega_2 \dot{A}_2 = -\epsilon\delta_2 A_1^2 e^{-i\epsilon\sigma t} + \epsilon^2 \bigg[& -2i\omega_2 D_1 B_2 - 2\delta_2 A_1 B_1 e^{-i\epsilon\sigma t} \\ & - \frac{\sigma\delta_2}{2\omega_2} A_1^2 e^{-i\epsilon\sigma t} + \frac{\delta_2^2}{\omega_1\omega_2}\left(1 - \frac{4\omega_1}{\omega_2 + 2\omega_1}\right) A_2 A_1 \bar{A}_1 \bigg] + \cdots \end{aligned} \tag{10.73}$$

When $B_1 = 0$ and $B_2 = 0$, (10.72) and (10.73) are not derivable from a Lagrangian because the coefficient of $A_1 A_2 \bar{A}_2$ in (10.72) is different from the coefficient of $A_2 A_1 \bar{A}_1$ in (10.73). The first is $-(3\delta_2^2)/(2\omega_1^2)$, whereas the second is 0 when $\omega_2 = 2\omega_1$. Moreover, the coefficient of $A_2 \bar{A}_1$ in (10.72) is not equal to twice the coefficient of A_1^2 in (10.73). To make these coefficients the same, we choose B_1 and B_2 in (10.70) and (10.71) such that

$$2i\omega_1 D_1 B_1 + D_1^2 A_1 = 0 \quad \text{and} \quad 2i\omega_2 D_1 B_2 + D_1^2 A_2 = 0 \tag{10.74}$$

or

$$B_1 = -\frac{\delta_2}{2\omega_1^2} A_2 \bar{A}_1 e^{i\sigma T_1} \quad \text{and} \quad B_2 = -\frac{\delta_2}{4\omega_2^2} A_1^2 e^{-i\sigma T_1} \tag{10.75}$$

Substituting (10.75) into (10.70) and (10.71), letting $\omega_2 = 2\omega_1$, and using the method of reconstitution yields

$$2i\omega_1 \dot{A}_1 = -2\epsilon\delta_2 A_2 \bar{A}_1 e^{i\epsilon\sigma t} + \epsilon^2 \left[\frac{9\delta_2^2}{8\omega_1^2} A_1^2 \bar{A}_1 + \frac{\delta_2^2}{2\omega_1^2} A_1 A_2 \bar{A}_2 \right] + \cdots \tag{10.76}$$

$$2i\omega_2 \dot{A}_2 = -\epsilon\delta_2 A_1^2 e^{-i\epsilon\sigma t} + \frac{\delta_2^2}{2\omega_1^2} \epsilon^2 A_2 A_1 \bar{A}_1 \tag{10.77}$$

which are derivable from the Lagrangian

$$L = i\omega_1 \left(A_1 \dot{\bar{A}}_1 - \bar{A}_1 \dot{A}_1 \right) + i\omega_2 \left(A_2 \dot{\bar{A}}_2 - \bar{A}_2 \dot{A}_2 \right) + \frac{9\delta_2^2}{16\omega_1^2} \epsilon^2 A_1^2 \bar{A}_1^2$$
$$- \epsilon\delta_2 \left(A_2 \bar{A}_1^2 e^{i\epsilon\sigma t} + A_1^2 \bar{A}_2 e^{-i\epsilon\sigma t} \right) + \frac{\delta_2^2}{2\omega_1^2} \epsilon^2 A_1 \bar{A}_1 A_2 \bar{A}_2 \tag{10.78}$$

Letting

$$\eta_m(t) = A_m(t) e^{i\omega_m t}$$

in (10.51) and (10.52) and setting $f_{1m} = 0$, $g_{1m} = 0$, $\delta_1 = 0$, $\delta_3 = 0$, and $\delta_4 = 0$, we obtain (10.76) and (10.77).

To compare the results obtained in this section with those obtained in Section 10.7, we express them in real-variable form by introducing the polar transformation

$$A_n = \tfrac{1}{2} a_n e^{i\beta_n}$$

into (10.76) and (10.77) and separating real and imaginary parts. The result is

$$\dot{a}_1 = -\frac{\epsilon\delta_2}{2\omega_1} a_1 a_2 \sin(\beta_2 - 2\beta_1 + \epsilon\sigma t) \tag{10.79}$$

$$\dot{a}_2 = \frac{\epsilon\delta_2}{4\omega_2} a_1^2 \sin(\beta_2 - 2\beta_1 + \epsilon\sigma t) \tag{10.80}$$

$$a_1 \dot{\beta}_1 = \frac{\epsilon\delta_2}{2\omega_1} a_1 a_2 \cos(\beta_2 - 2\beta_1 + \epsilon\sigma t) - \frac{9\epsilon^2\delta_2^2}{64\omega_1^3} a_1^3 - \frac{\epsilon^2\delta_2^2}{16\omega_1^3} a_1 a_2^2 \tag{10.81}$$

$$a_2 \dot{\beta}_2 = \frac{\epsilon\delta_2}{4\omega_2} a_1^2 \cos(\beta_2 - 2\beta_1 + \epsilon\sigma t) - \frac{\epsilon^2\delta_2^2}{32\omega_1^3} a_1^2 a_2 \tag{10.82}$$

Substituting (10.62), (10.68), and (10.69) into (10.55) and using (10.75) yields

$$u_1 = \left(A_1 - \frac{\epsilon\delta_2}{2\omega_1^2} A_2 \bar{A}_1 e^{i\sigma T_1}\right) e^{i\omega_1 T_0} + \frac{2\epsilon\delta_2}{\omega_2(\omega_2 + 2\omega_1)} A_2 A_1 e^{i(\omega_2+\omega_1)T_0} + \text{cc} \tag{10.83}$$

$$u_2 = \left(A_2 - \frac{\epsilon\delta_2}{4\omega_2^2} A_1^2 e^{-i\sigma T_1}\right) e^{i\omega_2 T_0} - \frac{\epsilon\delta_2}{\omega_2^2} A_1 \bar{A}_1 + \text{cc} \tag{10.84}$$

Then, using the polar transformation, we rewrite (10.83) and (10.84) as

$$u_1 = a_1 \cos(\omega_1 t + \beta_1) + \frac{\epsilon\delta_2}{\omega_2(\omega_2 + 2\omega_1)} a_1 a_2 \left\{ \cos[(\omega_2 + \omega_1)t + \beta_2 + \beta_1] - \frac{\omega_2(\omega_2 + 2\omega_1)}{4\omega_1^2} \cos[(\omega_2 - \omega_1)t + \beta_2 - \beta_1] \right\} + \cdots \tag{10.85}$$

$$u_2 = a_2 \cos(\omega_2 t + \beta_2) - \frac{\epsilon\delta_2 a_1^2}{8\omega_2^2} \{\cos(2\omega_1 t + 2\beta_1) + 4\} + \cdots \tag{10.86}$$

10.6.2
State-Space Form

We start with writing (10.53) and (10.54) in the state-space form

$$\dot{u}_1 - v_1 = 0 \tag{10.87}$$

$$\dot{v}_1 + \omega_1^2 u_1 = -2\epsilon\delta_2 u_1 u_2 \tag{10.88}$$

$$\dot{u}_2 - v_2 = 0 \tag{10.89}$$

$$\dot{v}_2 + \omega_2^2 u_2 = -\epsilon\delta_2 u_1^2 \tag{10.90}$$

Then, we seek a second-order approximate solution of (10.87)–(10.90) in the form

$$u_n = \sum_{m=0}^{2} \epsilon^m u_{nm}(T_0, T_1, T_2) + \cdots \tag{10.91}$$

$$v_n = \sum_{m=0}^{2} \epsilon^m v_{nm}(T_0, T_1, T_2) + \cdots \tag{10.92}$$

Substituting (10.91) into (10.87)–(10.90) and equating coefficients of like powers of ϵ yields.

Order (ϵ^0)

$$D_0 u_{10} - v_{10} = 0 \tag{10.93}$$

$$D_0 v_{10} + \omega_1^2 u_{10} = 0 \tag{10.94}$$

$$D_0 u_{20} - v_{20} = 0 \tag{10.95}$$

$$D_0 v_{20} + \omega_2^2 u_{20} = 0 \tag{10.96}$$

Order (ϵ)

$$D_0 u_{11} - v_{11} = -D_1 u_{10} \tag{10.97}$$

$$D_0 v_{11} + \omega_1^2 u_{11} = -D_1 v_{10} - 2\delta_2 u_{10} u_{20} \tag{10.98}$$

$$D_0 u_{21} - v_{21} = -D_1 u_{20} \tag{10.99}$$

$$D_0 v_{21} + \omega_2^2 u_{21} = -D_1 v_{20} - \delta_2 u_{10}^2 \tag{10.100}$$

Order (ϵ^2)

$$D_0 u_{12} - v_{12} = -D_1 u_{11} - D_2 u_{10} \tag{10.101}$$

$$D_0 v_{12} + \omega_1^2 u_{12} = -D_1 v_{11} - D_2 v_{10} - 2\delta_2 u_{10} u_{21} - 2\delta_2 u_{11} u_{20} \tag{10.102}$$

$$D_0 u_{22} - v_{22} = -D_1 u_{21} - D_2 u_{20} \tag{10.103}$$

$$D_0 v_{22} + \omega_2^2 u_{22} = -D_1 v_{21} - D_2 v_{20} - 2\delta_2 u_{10} u_{11} \tag{10.104}$$

The solutions of (10.93)–(10.96) can be expressed as

$$u_{10} = A_1(T_1, T_2)\, e^{i\omega_1 T_0} + \text{cc}\,, \quad v_{10} = i\omega_1 A_1(T_1, T_2)\, e^{i\omega_1 T_0} + \text{cc} \tag{10.105}$$

$$u_{20} = A_2(T_1, T_2)\, e^{i\omega_2 T_0} + \text{cc}\,, \quad v_{20} = i\omega_2 A_2(T_1, T_2)\, e^{i\omega_2 T_0} + \text{cc} \tag{10.106}$$

Substituting (10.105) and (10.106) into (10.97)–(10.100) yields

$$D_0 u_{11} - v_{11} = -D_1 A_1 e^{i\omega_1 T_0} + \text{cc} \tag{10.107}$$

$$\begin{aligned} D_0 v_{11} + \omega_1^2 u_{11} = & -i\omega_1 D_1 A_1 e^{i\omega_1 T_0} - 2\delta_2 A_2 A_1 e^{i(\omega_2+\omega_1)T_0} \\ & - 2\delta A_2 \bar{A}_1 e^{i(\omega_2-\omega_1)T_0} + \text{cc} \end{aligned} \tag{10.108}$$

$$D_0 u_{21} - v_{21} = -D_1 A_2 e^{i\omega_2 T_0} + \text{cc} \tag{10.109}$$

$$D_0 v_{21} + \omega_2^2 u_{21} = -i\omega_2 D_1 A_2 e^{i\omega_2 T_0} - \delta_2 A_1^2 e^{2i\omega_1 T_0} - \delta_2 A_1 \bar{A}_1 + \text{cc} \tag{10.110}$$

Because the homogeneous equations (10.107)–(10.110) have nontrivial solutions, the nonhomogeneous equations have solutions only if the nonhomogeneous terms are orthogonal to every solution of the corresponding adjoint equations. In this case, the solution of the adjoint of (10.107) and (10.108) is

$$[i\omega_1 \quad 1]^{\mathrm{T}} e^{-i\omega_1 T_0}$$

Therefore, using (10.65) and imposing the solvability condition on (10.107) and (10.108), one obtains

$$2i\omega_1 D_1 A_1 = -2\delta_2 A_2 \bar{A}_1 e^{i\sigma T_1} \tag{10.111}$$

Then, although (10.107) and (10.108) are solvable, their solution is not unique. A unique solution can be obtained by requiring it to be orthogonal to the above adjoint solution. Hence, the unique solution of (10.107) and (10.108) is

$$u_{11} = \frac{2\delta_2}{\omega_2(\omega_2 + 2\omega_1)} A_2 A_1 e^{i(\omega_2+\omega_1)T_0} - \frac{\delta_2}{2\omega_1^2} A_2 \bar{A}_1 e^{i(\omega_2-\omega_1)T_0} + \text{cc} \tag{10.112}$$

$$v_{11} = \frac{2i\delta_2(\omega_2 + \omega_1)}{\omega_2(\omega_2 + 2\omega_1)} A_2 A_1 e^{i(\omega_2+\omega_1)T_0} + \frac{i\delta_2}{\omega_1} A_2 \bar{A}_1 e^{i(\omega_2-\omega_1)T_0} + \text{cc} \tag{10.113}$$

Following steps similar to those used above, one finds that the solvability condition of (10.109) and (10.110) is

$$2i\omega_2 D_1 A_2 = -\delta_2 A_1^2 e^{-i\sigma T_1} \tag{10.114}$$

and their unique solution is

$$u_{21} = -\frac{\delta_2}{\omega_2^2} A_1 \bar{A}_1 - \frac{\delta_2}{4\omega_2^2} A_1^2 e^{2i\omega_1 T_0} + \text{cc} \tag{10.115}$$

$$v_{21} = \frac{i\delta_2}{4\omega_2} A_1^2 e^{2i\omega_1 T_0} + \text{cc} \tag{10.116}$$

Substituting (10.112), (10.113), (10.115), and (10.116) into (10.101)–(10.104) and imposing the solvability conditions yields

$$2i\omega_1 D_2 A_1 = \frac{\delta_2^2(8\omega_1 + 5\omega_2)}{\omega_2^2(\omega_2 + 2\omega_1)} A_1^2 \bar{A}_1 + \frac{2\delta_2^2}{\omega_1(\omega_2 + 2\omega_1)} A_1 A_2 \bar{A}_2 \tag{10.117}$$

$$2i\omega_2 D_2 A_2 = \frac{2\delta_2^2}{\omega_1(\omega_2 + 2\omega_1)} A_2 A_1 \bar{A}_1 \tag{10.118}$$

Using the method of reconstitution, we combine (10.111), (10.114), (10.117), and (10.118) into

$$2i\omega_1\dot{A}_1 = -2\epsilon\delta_2 A_2\bar{A}_1 e^{i\epsilon\sigma t} + \epsilon^2\left[\frac{\delta_2^2(8\omega_1+5\omega_2)}{\omega_2^2(\omega_2+2\omega_1)}A_1^2\bar{A}_1 + \frac{2\delta_2^2}{\omega_1(\omega_2+2\omega_1)}A_1A_2\bar{A}_2\right] + \cdots \tag{10.119}$$

$$2i\omega_2\dot{A}_2 = -\epsilon\delta_2 A_1^2 e^{-i\epsilon\sigma t} + \frac{2\epsilon^2\delta_2^2}{\omega_1(\omega_2+2\omega_1)}A_2A_1\bar{A}_1 + \cdots \tag{10.120}$$

Equations 10.119 and 10.120 are derivable from the Lagrangian

$$L = i\omega_1\left(A_1\dot{\bar{A}}_1 - \bar{A}_1\dot{A}_1\right) + i\omega_2\left(A_2\dot{\bar{A}}_2 - \bar{A}_2\dot{A}_2\right) - \epsilon\delta_2\left(A_2\bar{A}_1^2 e^{i\epsilon\sigma t} + A_1^2\bar{A}_2 e^{-i\epsilon\sigma t}\right) + \frac{\epsilon^2\delta_2^2(8\omega_1+5\omega_2)}{2\omega_2^2(\omega_2+2\omega_1)}A_1^2\bar{A}_1^2 + \frac{2\delta_2^2}{\omega_1(\omega_2+2\omega_1)}A_1\bar{A}_1A_2\bar{A}_2 \tag{10.121}$$

Equations 10.119–10.121 reduce to (10.76)–(10.78) when ω_2 is replaced with $2\omega_1$.

10.6.3 Complex-Valued Form

Using the transformation (10.4) and (10.5), we rewrite (10.53) and (10.54) in the complex-valued form

$$\dot{\zeta}_1 = i\omega_1\zeta_1 + \frac{i\epsilon\delta_2}{\omega_1}\left(\zeta_1+\bar{\zeta}_1\right)\left(\zeta_2+\bar{\zeta}_2\right) \tag{10.122}$$

$$\dot{\zeta}_2 = i\omega_2\zeta_2 + \frac{i\epsilon\delta_2}{2\omega_2}\left(\zeta_1+\bar{\zeta}_1\right)^2 \tag{10.123}$$

We seek a second-order uniform expansion of (10.122) and (10.123) in the form

$$\zeta_n(t;\epsilon) = \sum_{m=0}^{2}\epsilon^m\delta_{nm}\left(T_0, T_1, T_2\right) + \cdots \tag{10.124}$$

Substituting (10.124) into (10.122) and (10.123) and equating coefficients of like powers of ϵ, we obtain

Order (ϵ^0)

$$D_0\zeta_{11} - i\omega_1\zeta_{11} = 0 \tag{10.125}$$

$$D_0\zeta_{21} - i\omega_2\zeta_{21} = 0 \tag{10.126}$$

Order (ϵ)

$$D_0\zeta_{12} - i\omega_1\zeta_{12} = -D_1\zeta_{11} + \frac{i\delta_2}{\omega_1}\left(\zeta_{11} + \bar{\zeta}_{11}\right)\left(\zeta_{21} + \bar{\zeta}_{21}\right) \quad (10.127)$$

$$D_0\zeta_{22} - i\omega_2\zeta_{22} = -D_1\zeta_{21} + \frac{i\delta_2}{2\omega_2}\left(\zeta_{11} + \bar{\zeta}_{11}\right)^2 \quad (10.128)$$

Order (ϵ^2)

$$D_0\zeta_{13} - i\omega_1\zeta_{13} = -D_2\zeta_{11} - D_1\zeta_{12} + \frac{i\delta_2}{\omega_1}\left(\zeta_{11} + \bar{\zeta}_{11}\right)\left(\zeta_{22} + \bar{\zeta}_{22}\right) + \frac{i\delta_2}{\omega_1}\left(\zeta_{12} + \bar{\zeta}_{12}\right)\left(\zeta_{21} + \bar{\zeta}_{21}\right) \quad (10.129)$$

$$D_0\zeta_{23} - i\omega_2\zeta_{23} = -D_2\zeta_{21} - D_1\zeta_{22} + \frac{i\delta_2}{\omega_2}\left(\zeta_{11} + \bar{\zeta}_{11}\right)\left(\zeta_{12} + \bar{\zeta}_{12}\right) \quad (10.130)$$

The solutions of (10.125) and (10.126) can be expressed as

$$\zeta_{n1} = A_n\left(T_1, T_2\right)e^{i\omega_n T_0} \quad (10.131)$$

Substituting (10.131) into (10.127) and (10.128) yields

$$D_0\zeta_{12} - i\omega_1\zeta_{12} = -D_1A_1e^{i\omega_1 T_0} + \frac{i\delta_2}{\omega_1}\Big[A_2\bar{A}_1e^{i(\omega_2-\omega_1)T_0} + A_2A_1e^{i(\omega_2+\omega_1)T_0} + A_1\bar{A}_2e^{i(\omega_1-\omega_2)T_0} + \bar{A}_2\bar{A}_1e^{-i(\omega_2+\omega_1)T_0}\Big] \quad (10.132)$$

$$D_0\zeta_{22} - i\omega_2\zeta_{22} = -D_1A_2e^{i\omega_2 T_0} + \frac{i\delta_2}{2\omega_2}\left[A_1^2e^{2i\omega_1 T_0} + 2A_1\bar{A}_1 + \bar{A}_1^2e^{-2i\omega_1 T_0}\right] \quad (10.133)$$

Using (10.65) in (10.132) and (10.133) and eliminating the terms that produce secular terms, we obtain

$$D_1A_1 = \frac{i\delta_2}{\omega_1}A_2\bar{A}_1e^{i\sigma T_1} \quad (10.134)$$

$$D_1A_2 = \frac{i\delta_2}{2\omega_2}A_1^2e^{-i\sigma T_1} \quad (10.135)$$

Then, the solutions of (10.132) and (10.133) can be expressed as

$$\zeta_{12} = \frac{\delta_2}{\omega_1\omega_2}A_2A_1e^{i(\omega_2+\omega_1)T_0} - \frac{\delta_2}{\omega_1\omega_2}A_1\bar{A}_2e^{i(\omega_1-\omega_2)T_0} - \frac{\delta_2}{\omega_1(\omega_2+2\omega_1)}\bar{A}_2\bar{A}_1e^{-i(\omega_2+\omega_1)T_0} \quad (10.136)$$

$$\zeta_{22} = -\frac{\delta_2}{\omega_2^2}A_1\bar{A}_1 - \frac{\delta_2}{2\omega_2(\omega_2+2\omega_1)}\bar{A}_1^2e^{-i(\omega_2+\omega_1)T_0} \quad (10.137)$$

Substituting (10.131), (10.136), and (10.137) into (10.129) and (10.130), using (10.134) and (10.135), and eliminating the terms that lead to secular terms, we obtain

$$D_2 A_1 = -\frac{i\delta_2^2(5\omega_2 + 8\omega_1)}{2\omega_1\omega_2^2(\omega_2 + 2\omega_1)} A_1^2 \bar{A}_1 - \frac{i\delta_2^2}{\omega_1^2(\omega_2 + 2\omega_1)} \tag{10.138}$$

$$D_2 A_2 = -\frac{i\delta_2^2}{\omega_1\omega_2(\omega_2 + 2\omega_1)} A_2 A_1 \bar{A}_1 \tag{10.139}$$

Using the method of reconstitution, we combine (10.134), (10.135), (10.138), and (10.139) into

$$\dot{A}_1 = \frac{i\epsilon\delta_2}{\omega_1} A_2 \bar{A}_1 e^{i\sigma T_1} - \frac{i\epsilon^2\delta_2^2(5\omega_2 + 8\omega_1)}{2\omega_1\omega_2^2(\omega_2 + 2\omega_1)} A_1^2 \bar{A}_1 - \frac{i\epsilon^2\delta_2^2}{\omega_1^2(\omega_2 + 2\omega_1)} A_1 A_2 \bar{A}_2 \tag{10.140}$$

$$\dot{A}_2 = \frac{i\epsilon\delta_2}{2\omega_2} A_1^2 e^{-i\sigma T_1} - \frac{i\epsilon^2\delta_2^2}{\omega_1\omega_2(\omega_2 + 2\omega_1)} A_2 A_1 \bar{A}_1 \tag{10.141}$$

Equations 10.140 and 10.141 reduce to (10.76) and (10.77) when $\omega_2 = 2\omega_1$.

10.7
Generalized Method of Averaging

Using the method of variation of parameters, we transform (10.53) and (10.54) into

$$u_m = a_m(t)\cos\left[\phi_m(t)\right], \quad \dot{u}_m = -\omega_m a_m(t)\sin\left[\phi_m(t)\right] \tag{10.142}$$

where

$$\dot{a}_1 = -\frac{\epsilon\delta_2}{2\omega_1} a_1 a_2 \left[\sin(\phi_2 - 2\phi_1) - \sin(\phi_2 + 2\phi_1)\right] \tag{10.143}$$

$$\dot{\phi}_1 = \omega_1 + \frac{\epsilon\delta_2}{2\omega_1} a_2 \left[\cos(\phi_2 - 2\phi_1) + \cos(\phi_2 + 2\phi_1) + 2\cos\phi_2\right] \tag{10.144}$$

$$\dot{a}_2 = \frac{\epsilon\delta_2}{4\omega_2} a_1^2 \left[\sin(\phi_2 - 2\phi_1) + \sin(\phi_2 + 2\phi_1) + 2\sin\phi_2\right] \tag{10.145}$$

$$\dot{\phi}_2 = \omega_2 + \frac{\epsilon\delta_2 a_1^2}{4\omega_2 a_2} \left[\cos(\phi_2 - 2\phi_1) + \cos(\phi_2 + 2\phi_1) + 2\cos\phi_2\right] \tag{10.146}$$

We seek a second-order approximate solution of (10.143)–(10.146) in the form

$$a_m(t) = \eta_m(t) + \sum_{n=1}^{2} \epsilon^n a_{mn}(\psi_1, \psi_2, \eta_1, \eta_2) + \cdots \tag{10.147}$$

$$\phi_m(t) = \psi_m(t) + \sum_{n=1}^{2} \epsilon^n \phi_{mn}(\psi_1, \psi_2, \eta_1, \eta_2) + \cdots \tag{10.148}$$

where the a_{mn} and ϕ_{mn} are fast-varying functions of time,

$$\dot{\eta}_m = \epsilon A_{m1}(\eta_1, \eta_2, t) + \epsilon^2 A_{m2}(\eta_1, \eta_2, t) + \cdots \tag{10.149}$$

$$\dot{\psi}_m = \omega_m + \epsilon \Phi_{m1}(\eta_1, \eta_2, t) + \epsilon^2 \Phi_{m2}(\eta_1, \eta_2, t) + \cdots \tag{10.150}$$

and the A_{mn} and Φ_{mn} are slowly varying functions of time. Substituting (10.147)–(10.150) into (10.143)–(10.146) and equating the coefficients of ϵ to zero yields

$$A_{11} + \omega_1 \frac{\partial a_{11}}{\partial \psi_1} + \omega_2 \frac{\partial a_{11}}{\partial \psi_2} = -\frac{\delta_2}{2\omega_1}\eta_1\eta_2 [\sin(\psi_2 - 2\psi_1) - \sin(\psi_2 + 2\psi_1)] \tag{10.151}$$

$$\Phi_{11} + \omega_1 \frac{\partial \phi_{11}}{\partial \psi_1} + \omega_2 \frac{\partial \phi_{11}}{\partial \psi_2} = \frac{\delta_2}{2\omega_1}\eta_2[\cos(\psi_2 - 2\psi_1) + \cos(\psi_2 + 2\psi_1) + 2\cos\psi_2] \tag{10.152}$$

$$A_{21} + \omega_1 \frac{\partial a_{21}}{\partial \psi_1} + \omega_2 \frac{\partial a_{21}}{\partial \psi_2} = \frac{\delta_2}{4\omega_2}\eta_1^2[\sin(\psi_2 - 2\psi_1) + \sin(\psi_2 + 2\psi_1) + 2\sin\psi_2] \tag{10.153}$$

$$\Phi_{21} + \omega_1 \frac{\partial \phi_{21}}{\partial \psi_1} + \omega_2 \frac{\partial \phi_{21}}{\partial \psi_2} = \frac{\delta_2\eta_1^2}{4\omega_2\eta_2}[\cos(\psi_2 - 2\psi_1) + \cos(\psi_2 + 2\psi_1) + 2\cos\psi_2] \tag{10.154}$$

Choosing the A_{n1} and Φ_{n1} to eliminate the slowly varying terms in (10.151)–(10.154) and using (10.65), we obtain

$$A_{11} = -\frac{\delta_2}{2\omega_1}\eta_1\eta_2 \sin(\psi_2 - 2\psi_1) \tag{10.155}$$

$$\Phi_{11} = \frac{\delta_2}{2\omega_1}\eta_2 \cos(\psi_2 - 2\psi_1) \tag{10.156}$$

$$A_{21} = \frac{\delta_2}{4\omega_2}\eta_1^2 \sin(\psi_2 - 2\psi_1) \tag{10.157}$$

$$\Phi_{21} = \frac{\delta_2\eta_1^2}{4\omega_2\eta_2}\cos(\psi_2 - 2\psi_1) \tag{10.158}$$

Then, the solutions of (10.151)–(10.154) are given uniquely by

$$a_{11} = -\frac{\delta_2\eta_1\eta_2}{2\omega_1(\omega_2 + 2\omega_1)}\cos(\psi_2 + 2\psi_1) \tag{10.159}$$

$$\phi_{11} = \frac{\delta_2\eta_2}{2\omega_1(\omega_2 + 2\omega_1)}\sin(\psi_2 + 2\psi_1) + \frac{\delta_2\eta_2}{\omega_1\omega_2}\sin\psi_2 \tag{10.160}$$

$$a_{21} = -\frac{\delta_2\eta_1^2}{4\omega_2(\omega_2 + 2\omega_1)}\cos(\psi_2 + 2\psi_1) - \frac{\delta_2\eta_1^2}{2\omega_2^2}\cos\psi_2 \tag{10.161}$$

$$\phi_{21} = \frac{\delta_2\eta_1^2}{4\omega_2(\omega_2 + 2\omega_1)\eta_2}\sin(\psi_2 + 2\psi_1) + \frac{\delta_2\eta_1^2}{2\omega_2^2\eta_2}\sin\psi_2 \tag{10.162}$$

Substituting (10.155)–(10.162) into the second-order problem and choosing the A_{n2} and Φ_{n2} to eliminate the slowly varying terms in the resulting equations, we obtain

$$A_{12} = 0 \quad \text{and} \quad A_{22} = 0 \tag{10.163}$$

$$\Phi_{12} = -\frac{\delta_2^2}{8\omega_1^2\omega_2^2(\omega_2 + 2\omega_1)}\left[(8\omega_1^2 + 5\omega_1\omega_2)\eta_1^2 + 2\omega_2^2\eta_2^2\right] \tag{10.164}$$

$$\Phi_{22} = -\frac{\delta_2^2}{4\omega_1\omega_2(\omega_2 + 2\omega_1)}\eta_1^2 \tag{10.165}$$

Substituting (10.147) and (10.148) into (10.142), expanding the result for small ϵ, and using (10.159)–(10.162) yields

$$u_1 = \eta_1 \cos(\omega_1 t + \beta_1) - \frac{\epsilon\delta_2}{2\omega_1\omega_2}\eta_1\eta_2 \cos[(\omega_2 - \omega_1)t + \beta_2 - \beta_1] + \frac{\epsilon\delta_2\eta_1\eta_2}{\omega_2(\omega_2 + 2\omega_1)}\cos[(\omega_2 + \omega_1)t + \beta_2 + \beta_1] + \cdots \tag{10.166}$$

$$u_2 = \eta_2 \cos(\omega_2 t + \beta_2) - \frac{\epsilon\delta_2\eta_1^2}{4\omega_2(\omega_2 + 2\omega_1)}\cos(2\omega_1 t + 2\beta_1) - \frac{\epsilon\delta_2}{2\omega_2^2}\eta_1^2 + \cdots \tag{10.167}$$

Substituting (10.155)–(10.158) and (10.163)–(10.165) into (10.149) and (10.150), we obtain the modulation equations

$$\dot{\eta}_1 = -\frac{\epsilon\delta_2}{2\omega_1}\eta_1\eta_1 \sin[(\omega_2 - 2\omega_1)t + \beta_2 - 2\beta_1] + O(\epsilon^3) \tag{10.168}$$

$$\dot{\psi}_1 = \omega_1 + \frac{\epsilon\delta_2}{2\omega_1}\eta_2 \cos[(\omega_2 - 2\omega_1)t + \beta_2 - 2\beta_1] - \frac{\epsilon^2\delta_2^2}{8\omega_1^2\omega_2^2(\omega_2 + 2\omega_1)}\left[(8\omega_1^2 + 5\omega_1\omega_2)\eta_1^2 + 2\omega_2^2\eta_2^2\right] + O(\epsilon^3) \tag{10.169}$$

$$\dot{\eta}_2 = \frac{\epsilon\delta_2}{4\omega_2}\eta_1^2 \sin[(\omega_2 - 2\omega_1)t + \beta_2 - 2\beta_1] + O(\epsilon^3) \tag{10.170}$$

$$\dot{\psi}_2 = \omega_2 + \frac{\epsilon\delta_2\eta_1^2}{4\omega_2\eta_2}\cos[(\omega_2 - 2\omega_1)t + \beta_2 - 2\beta_1] - \frac{\epsilon^2\delta_2^2}{4\omega_1\omega_2(\omega_2 + 2\omega_1)}\eta_1^2 + O(\epsilon^3) \tag{10.171}$$

Equations 10.166–10.171 are in full agreement with (10.85), (10.86), and (10.79)–(10.82) obtained with the method of multiple scales because $\omega_2 \approx 2\omega_1$, $\eta_n = a_n$, and $\psi_n = \omega_n t + \beta_n$.

10.8 A Nonsemisimple One-to-One Internal Resonance

In this section, we use the methods of normal form and multiple scales to treat a two-degree-of-freedom system with quadratic and cubic nonlinearities whose linear part has a generic nonsemisimple structure. Specifically, we treat the system

$$\ddot{u}_1 + \omega^2 u_1 + 2\mu_1 \dot{u}_1 + \delta_{11} u_1^2 + \delta_{12} u_1 u_2 + \delta_{13} u_2^2 + \alpha_{11} u_1^3 + \alpha_{12} u_1^2 u_2 + \alpha_{13} u_1 u_2^2 + \alpha_{14} u_2^3 = 0 \tag{10.172}$$

$$\ddot{u}_2 + \omega^2 u_2 + 2\mu_2 \dot{u}_2 + u_1 + \delta_{21} u_1^2 + \delta_{22} u_1 u_2 + \delta_{23} u_2^2 + \alpha_{21} u_1^3 + \alpha_{22} u_1^2 u_2 + \alpha_{23} u_1 u_2^2 + \alpha_{24} u_2^3 = 0 \tag{10.173}$$

10.8.1 The Method of Normal Forms

Again, as a first step, we use the transformation (9.37) and (9.38) to recast (10.172) and (10.173) in the complex-valued form

$$\begin{aligned}\dot{\zeta}_1 = {} & i\omega\zeta_1 - \mu_1\left(\zeta_1 - \bar{\zeta}_1\right) + \frac{i}{2\omega}\Big[\delta_{11}\left(\zeta_1 + \bar{\zeta}_1\right)^2 + \delta_{12}\left(\zeta_1 + \bar{\zeta}_1\right)\left(\zeta_2 + \bar{\zeta}_2\right) \\ & + \delta_{13}\left(\zeta_2 + \bar{\zeta}_2\right)^2\Big] + \frac{i}{2\omega}\Big[\alpha_{11}\left(\zeta_1 + \bar{\zeta}_1\right)^3 + \alpha_{12}\left(\zeta_1 + \bar{\zeta}_1\right)^2\left(\zeta_2 + \bar{\zeta}_2\right) \\ & + \alpha_{13}\left(\zeta_1 + \bar{\zeta}_1\right)\left(\zeta_2 + \bar{\zeta}_2\right)^2 + \alpha_{14}\left(\zeta_2 + \bar{\zeta}_2\right)^3\Big]\end{aligned} \tag{10.174}$$

$$\begin{aligned}\dot{\zeta}_2 = {} & i\omega\zeta_2 - \mu_2\left(\zeta_2 - \bar{\zeta}_2\right) + \frac{i}{2\omega}\left(\zeta_1 + \bar{\zeta}_1\right) + \frac{i}{2\omega}\Big[\delta_{21}\left(\zeta_1 + \bar{\zeta}_1\right)^2 \\ & + \delta_{22}\left(\zeta_1 + \bar{\zeta}_1\right)\left(\zeta_2 + \bar{\zeta}_2\right) + \delta_{23}\left(\zeta_2 + \bar{\zeta}_2\right)^2\Big] + \frac{i}{2\omega}\Big[\alpha_{21}\left(\zeta_1 + \bar{\zeta}_1\right)^3 \\ & + \alpha_{22}\left(\zeta_1 + \bar{\zeta}_1\right)^2\left(\zeta_2 + \bar{\zeta}_2\right) + \alpha_{23}\left(\zeta_1 + \bar{\zeta}_1\right)\left(\zeta_2 + \bar{\zeta}_2\right)^2 \\ & + \alpha_{24}\left(\zeta_2 + \bar{\zeta}_2\right)^3\Big]\end{aligned} \tag{10.175}$$

To simplify the algebra, we scale the variables in (10.174) and (10.175) before determining their normal form. We treat the case of weak nonlinearities and damping. Thus, we assume that $\zeta_1 = O(\epsilon)$, where ϵ is a small nondimensional parameter that is used as a bookkeeping device. Because ζ_2 is much larger than ζ_1 due to the nonsemisimple structure of the linear undamped operator in (10.172) and (10.173), $\zeta_2 = O(\epsilon^{1-\lambda_2})$, where $\lambda_2 > 0$. Because the damping is weak, $\mu_n = O(\epsilon^{\lambda_1})$, where $\lambda_1 > 0$. To make these different orderings explicit, we introduce new scaled variables defined by

$$\zeta_1 = \epsilon\eta_1 , \quad \zeta_2 = \epsilon^{1-\lambda_2}\eta_2 , \quad \mu_n = \epsilon^{\lambda_1}\hat{\mu}_n$$

where the η_n and $\hat{\mu}_n$ are $O(1)$ and rewrite (10.174) and (10.175) as

$$\begin{aligned}\dot{\eta}_1 &= i\omega\eta_1 - \epsilon^{\lambda_1}\hat{\mu}_1(\eta_1 - \bar{\eta}_1) + \frac{i}{2\omega}\Big[\epsilon\delta_{11}(\eta_1+\bar{\eta}_1)^2 \\ &\quad + \epsilon^{1-\lambda_2}\delta_{12}(\eta_1+\bar{\eta}_1)(\eta_2+\bar{\eta}_2) + \epsilon^{1-2\lambda_2}\delta_{13}(\eta_2+\bar{\eta}_2)^2\Big] \\ &\quad + \frac{i}{2\omega}\Big[\epsilon^2\alpha_{11}(\eta_2+\bar{\eta}_1)^3 + \epsilon^{2-\lambda_2}\alpha_{12}(\eta_1+\bar{\eta}_1)^2(\eta_2+\bar{\eta}_2) \\ &\quad + \epsilon^{2-2\lambda_2}\alpha_{13}(\eta_1+\bar{\eta}_1)(\eta_2+\bar{\eta}_2)^2 + \epsilon^{2-3\lambda_2}\alpha_{14}(\eta_2+\bar{\eta}_2)^3\Big]\end{aligned} \tag{10.176}$$

$$\begin{aligned}\dot{\eta}_2 &= i\omega\eta_2 - \epsilon^{\lambda_1}\hat{\mu}_2(\eta_2 - \bar{\eta}_2) \\ &\quad + \frac{i}{2\omega}\epsilon^{\lambda_2}(\eta_1+\bar{\eta}_1) + \frac{i}{2\omega}\Big[\epsilon^{1+\lambda_2}\delta_{21}(\eta_1+\bar{\eta}_1)^2 \\ &\quad + \epsilon\delta_{22}(\eta_1+\bar{\eta}_1)(\eta_2+\bar{\eta}_2) + \epsilon^{1-\lambda_2}\delta_{23}(\eta_2+\bar{\eta}_2)^2\Big] \\ &\quad + \frac{i}{2\omega}\Big[\epsilon^{2+\lambda_2}\alpha_{21}(\eta_1+\bar{\eta}_1)^3 + \epsilon^2\alpha_{22}(\eta_1+\bar{\eta}_1)^2(\eta_2+\bar{\eta}_2) \\ &\quad + \epsilon^{2-\lambda_2}\alpha_{23}(\eta_1+\bar{\eta}_1)(\eta_2+\bar{\eta}_2)^2 + \epsilon^{2-2\lambda_2}\alpha_{24}(\eta_2+\bar{\eta}_2)^3\Big]\end{aligned} \tag{10.177}$$

Because of the one-to-one internal resonance, $\eta_1\bar{\eta}_1\eta_2$, $\eta_2\bar{\eta}_2\eta_1$, $\eta_1^2\bar{\eta}_1$, $\eta_2^2\bar{\eta}_2$, $\eta_1^2\bar{\eta}_2$, $\eta_2^2\bar{\eta}_1$, η_1, and η_2 are resonance terms in (10.176) and (10.177). Balancing the damping term and the dominant resonance term in (10.176), namely, the term proportional to α_{14}, we have

$$\lambda_1 = 2 - 3\lambda_2 \tag{10.178}$$

Balancing the damping term and the dominant resonance term in (10.177), namely, the term $i(2\omega)^{-1}\eta_1$, we have

$$\lambda_1 = \lambda_2 \tag{10.179}$$

Solving (10.178) and (10.179) yields

$$\lambda_1 = \lambda_2 = \tfrac{1}{2} \tag{10.180}$$

Hence, (10.176) and (10.177) become

$$\begin{aligned}\dot{\eta}_1 &= i\omega\eta_1 + \frac{i}{2\omega}\delta_{13}(\eta_2+\bar{\eta}_2)^2 - \epsilon^{1/2}\hat{\mu}_1(\eta_1-\bar{\eta}_1) \\ &\quad + \frac{i\epsilon^{1/2}}{2\omega}\Big[\delta_{12}(\eta_1+\bar{\eta}_1)(\eta_2+\bar{\eta}_2) + \alpha_{14}(\eta_2+\bar{\eta}_2)^3\Big] + \cdots\end{aligned} \tag{10.181}$$

$$\dot{\eta}_2 = i\omega\eta_2 - \epsilon^{1/2}\hat{\mu}_2(\eta_2-\bar{\eta}_2) + \frac{i\epsilon^{1/2}}{2\omega}\Big[\eta_1+\bar{\eta}_1+\delta_{23}(\eta_2+\bar{\eta}_2)^3\Big] + \cdots \tag{10.182}$$

To simplify (10.181) and (10.182), we introduce the transformation

$$\eta_1 = \xi_1 + h_{11}\left(\xi_n, \bar{\xi}_n\right) + \epsilon^{1/2} h_{12}\left(\xi_n, \bar{\xi}_n\right) + \cdots \tag{10.183}$$

$$\eta_2 = \xi_2 + \epsilon^{1/2} h_{22}\left(\xi_n, \bar{\xi}_n\right) + \cdots \tag{10.184}$$

and choose the h_{mn} to eliminate the nonresonance terms so that the resulting equations have the simplest possible form

$$\dot{\xi}_n = i\omega\xi_n + \epsilon^{1/2} g_n\left(\xi_m, \bar{\xi}_m\right) + \cdots \tag{10.185}$$

Substituting (10.183)–(10.185) into (10.181) and (10.182) and equating coefficients of like powers of ϵ, we obtain

$$\mathcal{L}(h_{11}) = \frac{i}{2\omega}\delta_{13}\left(\xi_2 + \bar{\xi}_2\right)^2 \tag{10.186}$$

$$\begin{aligned} g_1 + \mathcal{L}(h_{12}) = & -g_1\frac{\partial h_{11}}{\partial \xi_1} - \bar{g}_1\frac{\partial h_{11}}{\partial \bar{\xi}_1} - g_2\frac{\partial h_{11}}{\partial \xi_2} - \bar{g}_2\frac{\partial h_{11}}{\partial \bar{\xi}_2} \\ & - \hat{\mu}_1\left(\xi_1 - \bar{\xi}_1 + h_{11} - \bar{h}_{11}\right) + \frac{i}{\omega}\delta_{13}\left(\xi_2 + \bar{\xi}_2\right)\left(h_{22} + \bar{h}_{22}\right) \\ & + \frac{i}{2\omega}\left[\delta_{12}\left(\xi_1 + \bar{\xi}_1 + h_{11} + \bar{h}_{11}\right)\left(\xi_2 + \bar{\xi}_2\right) + \alpha_{14}\left(\xi_2 + \bar{\xi}_2\right)^3\right] \end{aligned} \tag{10.187}$$

$$\begin{aligned} g_2 + \mathcal{L}(h_{22}) = & -\hat{\mu}_2\left(\xi_2 - \bar{\xi}_2\right) + \frac{i}{2\omega}\left[\xi_1 + \bar{\xi}_1 + h_{11} + \bar{h}_{11}\right] \\ & + \frac{i}{2\omega}\delta_{23}\left(\xi_2 + \bar{\xi}_2\right)^2 \end{aligned} \tag{10.188}$$

where

$$\mathcal{L}(h) = i\omega\left[\frac{\partial h}{\partial \xi_1}\xi_1 - \frac{\partial h}{\partial \bar{\xi}_1}\bar{\xi}_1 + \frac{\partial h}{\partial \xi_2}\xi_2 - \frac{\partial h}{\partial \bar{\xi}_2}\bar{\xi}_2 - h\right]$$

The right-hand side of (10.186) suggests seeking h_{11} in the form

$$h_{11} = \Gamma_1\xi_2^2 + \Gamma_2\xi_2\bar{\xi}_2 + \Gamma_3\bar{\xi}_2^2 \tag{10.189}$$

Substituting (10.189) into (10.186) and equating each of the coefficients of ξ_2^2, $\xi_2\bar{\xi}_2$, and $\bar{\xi}_2^2$ on both sides, we obtain

$$\Gamma_1 = \frac{\delta_{13}}{2\omega^2}, \ \Gamma_2 = -\frac{\delta_{13}}{\omega^2}, \ \Gamma_3 = -\frac{\delta_{13}}{6\omega^2} \tag{10.190}$$

It follows from (10.189) and (10.190) that

$$h_{11} + \bar{h}_{11} = \frac{\delta_{13}}{3\omega^2}\left(\xi_2^2 + \bar{\xi}_2^2 - 6\xi_2\bar{\xi}_2\right) \tag{10.191}$$

Substituting (10.191) into (10.188) yields

$$g_2 + i\omega\left[\frac{\partial h_{22}}{\partial \xi_1}\xi_1 - \frac{\partial h_{22}}{\partial \bar{\xi}_1}\bar{\xi}_1 + \frac{\partial h_{22}}{\partial \xi_2}\xi_2 - \frac{\partial h_{22}}{\partial \bar{\xi}_2}\bar{\xi}_2 - h_{22}\right] = -\hat{\mu}_2\left(\xi_2 - \bar{\xi}_2\right)$$
$$+ \frac{i}{2\omega}\left[\xi_1 + \bar{\xi}_1 + \frac{\delta_{13}}{3\omega^2}\left(\xi_2^2 + \bar{\xi}_2^2 - 6\xi_2\bar{\xi}_2\right) + \delta_{23}\left(\xi_2 + \bar{\xi}_2\right)^2\right] \quad (10.192)$$

Equating g_2 to the resonance terms in (10.192), we have

$$g_2 = -\hat{\mu}_2\xi_2 + \frac{i}{2\omega}\xi_1 \quad (10.193)$$

Then, substituting

$$h_{22} = \Gamma_4\bar{\xi}_2 + \Gamma_5\bar{\xi}_1 + \Gamma_6\xi_2^2 + \Gamma_7\xi_2\bar{\xi}_2 + \Gamma_8\bar{\xi}_2^2 \quad (10.194)$$

into (10.192), using (10.193), and equating each of the coefficients of $\bar{\xi}_2$, $\bar{\xi}_1$, ξ_2^2, $\xi_2\bar{\xi}_2$, and $\bar{\xi}_2^2$ on both sides, we obtain

$$\Gamma_4 = \frac{i\hat{\mu}_2}{2\omega}, \quad \Gamma_5 = -\frac{1}{4\omega^2}, \quad \Gamma_6 = \frac{\delta_{23}}{2\omega^2} + \frac{\delta_{13}}{6\omega^4},$$
$$\Gamma_7 = -\frac{\delta_{23}}{\omega^2} + \frac{\delta_{13}}{\omega^4}, \quad \Gamma_8 = -\frac{\delta_{23}}{6\omega^2} - \frac{\delta_{13}}{18\omega^4} \quad (10.195)$$

Substituting (10.189)–(10.191) and (10.193)–(10.195) into (10.187) yields

$$g_1 + i\omega\left[\frac{\partial h_{12}}{\partial \xi_1}\xi_1 - \frac{\partial h_{12}}{\partial \bar{\xi}_1}\bar{\xi}_1 + \frac{\partial h_{12}}{\partial \xi_2}\xi_2 - \frac{\partial h_{12}}{\partial \bar{\xi}_2}\bar{\xi}_2 - h_{12}\right]$$
$$= -\left(2\Gamma_1\xi_2 + \Gamma_2\bar{\xi}_2\right)\left(-\hat{\mu}_2\xi_2 + \frac{i}{2\omega}\xi_1\right)$$
$$- \left(\Gamma_2\xi_2 + 2\Gamma_3\bar{\xi}_2\right)\left(-\hat{\mu}_2\bar{\xi}_2 - \frac{i}{2\omega}\bar{\xi}_1\right)$$
$$- \hat{\mu}_1\left[\xi_1 - \bar{\xi}_1 + \frac{2\delta_{13}}{3\omega^2}\left(\xi_2^2 - \bar{\xi}_2^2\right)\right]$$
$$+ \frac{i}{\omega}\delta_{13}\left(\xi_2 + \bar{\xi}_2\right)\left[-\frac{i\hat{\mu}_2}{2\omega}\left(\xi_2 - \bar{\xi}_2\right) - \frac{1}{4\omega^2}\left(\xi_1 + \bar{\xi}_1\right)\right.$$
$$\left.+ \frac{\delta_{13}}{9\omega^4}\left(\xi_2^2 + \bar{\xi}_2^2 + 18\xi_2\bar{\xi}_2\right) + \frac{\delta_{23}}{3\omega^2}\left(\xi_2^2 + \bar{\xi}_2^2 - 6\xi_2\bar{\xi}_2\right)\right]$$
$$+ \frac{i}{2\omega}\left[\delta_{12}\left(\xi_1 + \bar{\xi}_1\right)\left(\xi_2 + \bar{\xi}_2\right) + \alpha_{14}\left(\xi_2 + \bar{\xi}_2\right)^3\right.$$
$$\left.+ \frac{\delta_{12}\delta_{13}}{3\omega^2}\left(\xi_2^2 + \bar{\xi}_2^2 - 6\xi_2\bar{\xi}_2\right)\left(\xi_2 + \bar{\xi}_2\right)\right] \quad (10.196)$$

Since we are stopping at this order, we do not need to determine h_{12} explicitly, and hence all that we need to do is to determine g_1. Choosing g_1 to eliminate the resonance terms from (10.196), we have

$$g_1 = -\hat{\mu}_1\xi_1 + i\alpha_e\xi_2^2\bar{\xi}_2 \quad (10.197)$$

where

$$\alpha_e = \frac{1}{2\omega}\left[3\alpha_{14} - \frac{10}{3\omega^2}\delta_{13}\delta_{23} - \frac{5}{3\omega^2}\delta_{12}\delta_{13} + \frac{38}{9\omega^4}\delta_{13}^2\right] \tag{10.198}$$

Substituting for the g_n and h_{mn} in (10.183)–(10.185), we obtain

$$\eta_1 = \xi_1 + \frac{\delta_{13}}{6\omega^2}\left(3\xi_2^2 - 6\xi_2\bar{\xi}_2 - \bar{\xi}_2^2\right) + \cdots \tag{10.199}$$

$$\eta_2 = \xi_2 + \cdots \tag{10.200}$$

$$\dot{\xi}_1 = i\omega\xi_1 - \epsilon^{1/2}\hat{\mu}_1\xi_1 + i\epsilon^{1/2}\alpha_e\xi_2^2\bar{\xi}_2 + \cdots \tag{10.201}$$

$$\dot{\xi}_2 = i\omega\xi_2 - \epsilon^{1/2}\hat{\mu}_2\xi_2 + \frac{i\epsilon^{1/2}}{2\omega}\xi_1 + \cdots \tag{10.202}$$

10.8.2
The Method of Multiple Scales

Alternatively, we use the method of multiple scales to determine an approximation to the solution of (10.172) and (10.173). Following a procedure similar to that used above, we note that if $u_1 = O(\epsilon)$, then $u_2 = O(\epsilon^{1/2})$ and $\mu_n = O(\epsilon^{1/2})$. Hence, we introduce scaled variables defined by

$$u_1 = \epsilon v_1\,, \quad u_2 = \epsilon^{1/2}v_2\,, \quad \mu_n = \epsilon^{1/2}\hat{\mu}_n \tag{10.203}$$

in (10.172) and (10.173) and obtain

$$\ddot{v}_1 + \omega^2 v_1 + \delta_{13}v_2^2 + 2\epsilon^{1/2}\hat{\mu}_1\dot{v}_1 + \epsilon^{1/2}\left(\delta_{12}v_1v_2 + \alpha_{14}v_2^3\right) + \cdots = 0 \tag{10.204}$$

$$\ddot{v}_2 + \omega^2 v_2 + 2\epsilon^{1/2}\hat{\mu}_2\dot{v}_2 + \epsilon^{1/2}\left(v_1 + \delta_{23}v_2^2\right) + \cdots = 0 \tag{10.205}$$

We seek an expansion of the solution of (10.204) and (10.205) valid up to $O(\epsilon^{1/2})$ in the form

$$v_1(t;\epsilon) = v_{10}(T_0, T_1) + \epsilon^{1/2}v_{11}(T_0, T_1) + \cdots \tag{10.206}$$

$$v_2(t;\epsilon) = v_{20}(T_0, T_1) + \epsilon^{1/2}v_{21}(T_0, T_1) + \cdots \tag{10.207}$$

where $T_0 = t$ and $T_1 = \epsilon^{1/2}t$. In terms of T_0 and T_1, the time derivatives become

$$\frac{d}{dt} = D_0 + \epsilon^{1/2}D_1 + \cdots \quad \text{and} \quad \frac{d^2}{dt^2} = D_0^2 + 2\epsilon^{1/2}D_0D_1 + \cdots$$

where $D_n = \partial/\partial T_n$. Substituting (10.206) and (10.207) into (10.204) and (10.205) and equating coefficients of like powers of ϵ, we obtain

Order (ϵ^0)

$$D_0^2 \nu_{10} + \omega^2 \nu_{10} = -\delta_{13} \nu_{20}^2 \tag{10.208}$$

$$D_0^2 \nu_{20} + \omega^2 \nu_{20} = 0 \tag{10.209}$$

Order (ϵ)

$$D_0^2 \nu_{11} + \omega^2 \nu_{11} = -2 D_0 D_1 \nu_{10} - 2\hat{\mu}_1 D_0 \nu_{10} - 2\delta_{13} \nu_{20} \nu_{21} - \delta_{12} \nu_{10} \nu_{20} - \alpha_{14} \nu_{20}^3 \tag{10.210}$$

$$D_0^2 \nu_{21} + \omega^2 \nu_{21} = -2 D_0 D_1 \nu_{20} - 2\hat{\mu}_2 D_0 \nu_{20} - \nu_{10} - \delta_{23} \nu_{20}^2 \tag{10.211}$$

The solution of (10.209) can be expressed as

$$\nu_{20} = A_2(T_1) e^{i\omega T_0} + \text{cc} \tag{10.212}$$

Then, (10.208) becomes

$$D_0^2 \nu_{10} + \omega^2 \nu_{10} = -\delta_{13} \left(A_2^2 e^{2i\omega T_0} + A_2 \bar{A}_2 \right) + \text{cc}$$

whose solution can be expressed as

$$\nu_{10} = A_1(T_1) e^{i\omega T_0} + \frac{\delta_{13}}{3\omega^2} A_2^2 e^{2i\omega T_0} - \frac{\delta_{13}}{\omega^2} A_2 \bar{A}_2 + \text{cc} \tag{10.213}$$

Substituting (10.212) and (10.213) into (10.210) and (10.211) yields

$$D_0^2 \nu_{11} + \omega^2 \nu_{11} = -2i\omega \left(A_1' + \hat{\mu}_1 A_1 \right) e^{i\omega T_0} - \left(3\alpha_{14} - \frac{5}{3\omega^2} \delta_{12} \delta_{13} \right) A_2^2 \bar{A}_2 e^{i\omega T_0} + \text{cc} + \text{NST} - 2\delta_{13} \nu_{20} \nu_{21} \tag{10.214}$$

$$D_0^2 \nu_{21} + \omega^2 \nu_{21} = -2i\omega \left(A_2' + \hat{\mu}_2 A_2 \right) e^{i\omega T_0} - A_1 e^{i\omega T_0} - \frac{\delta_{13}}{3\omega^2} A_2^2 e^{2i\omega T_0} + \frac{\delta_{13}}{\omega^2} A_2 \bar{A}_2 - \delta_{23} \left(A_2^2 e^{2i\omega T_0} + A_2 \bar{A}_2 \right) + \text{cc} \tag{10.215}$$

Eliminating the term that produces secular terms from (10.215), we have

$$2i\omega \left(A_2' + \hat{\mu}_2 A_2 \right) + A_1 = 0 \tag{10.216}$$

Then, the solution of (10.215) can be expressed as

$$\nu_{21} = \left(\frac{\delta_{13}}{9\omega^4} + \frac{\delta_{23}}{3\omega^2} \right) A_2^2 e^{2i\omega T_0} + \left(\frac{\delta_{13}}{\omega^4} - \frac{\delta_{23}}{\omega^2} \right) A_2 \bar{A}_2 + \text{cc} \tag{10.217}$$

Substituting (10.217) into (10.214) and eliminating the terms that produce secular terms, we obtain

$$2i\omega \left(A_1' + \hat{\mu}_1 A_1 \right) + 2\omega \alpha_e A_2^2 \bar{A}_2 = 0 \tag{10.218}$$

where α_e is given by (10.198).

Letting $\xi_n = A_n e^{i\omega_n t}$ in (10.201) and (10.202) yields (10.216) and (10.218). Hence, the results obtained by using the method of normal forms are in full agreement with those found by using the method of multiple scales.

10.9 Exercises

10.9.1 Consider the system

$$\ddot{u}_1 + \omega_1^2 u_1 = \delta_1 \dot{u}_1 \dot{u}_2$$
$$\ddot{u}_2 + \omega_2^2 u_2 = \delta_2 \dot{u}_1^2$$

Use the method of normal forms, the method of multiple scales, and the generalized method of averaging to determine a second-order approximation to the solution of this system when $\omega_2 \approx 2\omega_1$.

10.9.2 Consider the system

$$\ddot{u}_1 + \omega_1^2 u_1 = \delta_1 u_1 \ddot{u}_2$$
$$\ddot{u}_2 + \omega_2^2 u_2 = \delta_2 u_1 \ddot{u}_1$$

Use the method of normal forms, the method of multiple scales, and the generalized method of averaging to determine a second-order approximation to the solution of this system when $\omega_2 \approx 2\omega_1$.

11
Retarded Systems

Several approaches have been proposed in the literature to analyze the nature of Hopf bifurcations of retarded systems, including integral averaging, the Fredholm alternative, the implicit function theorem, the method of multiple scales, and center-manifold reduction. In this chapter, we follow Nayfeh (2008) and use two of these approaches, the method of multiple scales and center-manifold reduction, to analyze the nature of Hopf bifurcations in retarded systems modeled by nonlinear homogeneous ordinary-differential equations with discrete time delay. Such equations model the behavior of many physical systems arising in physiology, biology, population dynamics, machine-tool chatter, neural networks, and time-delayed feedback controlled systems.

To describe the approaches without getting involved in the algebra, we use three simple systems, namely a scalar equation, a single-degree-of-freedom system, and a three-neuron model. We show that the normal forms obtained with all the approaches are the same. However, the method of multiple scales seems to be simpler. In fact, the method of multiple scales is directly applied to the retarded differential equations. In contrast, in the center-manifold approach, one needs to convert the retarded differential equations into operator equations in a Banach space, introduce a device that acts like an inner product because the Banach space does not have a natural inner product associated with its norm, define the adjoint associated with the linear part of the differential equations, perform the projection onto the center manifold, and calculate the normal form of the subsystem describing the dynamics on the center manifold. Finally, we consider a problem in which the retarded term appears as an acceleration and treat it using the method of multiple scales only.

11.1
A Scalar Equation

We start by a scalar equation with discrete time delay, which models a single neuron with a general activation function; that is,

$$\frac{d\hat{x}(t)}{dt} = -\hat{x}(t) + g\left[\hat{x}(t) - \beta\hat{x}(t-\tau)\right] \tag{11.1}$$

The Method of Normal Forms, Second Edition. Ali Hasan Nayfeh

where $g(\hat{x})$ is a general three-times continuously differentiable function, $\tau > 0$, and $\beta > 0$.

We let x^* denote an equilibrium of (11.1); that is,

$$x^* = g[(1-\beta)x^*]$$

Then, we let $\hat{x}(t) = x^* + x(t)$ in (11.1), expand the result in a Taylor series in x, keep up to cubic terms, and obtain

$$\dot{x}(t) = (\alpha_1 - 1)x(t) - \alpha_1 \beta x(t-\tau) - \alpha_2[x(t) - \beta x(t-\tau)]^2 \\ - \alpha_3 \left[x(t) - \beta x(t-\tau)\right]^3 \tag{11.2}$$

where

$$\alpha_1 = g'[(1-\beta)x^*], \quad \alpha_2 = -\tfrac{1}{2}g''[(1-\beta)x^*], \quad \alpha_3 = -\tfrac{1}{6}g'''[(1-\beta)x^*]$$

Linearizing (11.2) yields

$$\dot{x}(t) = (\alpha_1 - 1)x(t) - \alpha_1 \beta x(t-\tau) \tag{11.3}$$

which has solutions of the form

$$x = x_0 e^{(\sigma + i\omega)t} \tag{11.4}$$

where σ is the growth or decay rate, ω is the frequency of oscillations, and x_0 depends on the initial conditions. Substituting (11.4) into (11.3) yields the characteristic equation

$$\sigma + i\omega = \alpha_1 - 1 - \alpha_1 \beta e^{-(\sigma + i\omega)\tau} \tag{11.5}$$

For a given α_1, β, and τ, the complex-valued (11.5) has infinitely many solutions σ and ω.

When $\sigma > 0$ the trivial solution is unstable; when $\sigma < 0$ the trivial solution is asymptotically stable; and $\sigma = 0$ defines the stability boundary. To locate this boundary, we let $\sigma = 0$ in (11.5) and obtain the complex-valued characteristic equation

$$i\omega = \alpha_1 - 1 - \alpha_1 \beta e^{-i\omega\tau}$$

which, upon separating its real and imaginary parts, yields

$$\omega = \alpha_1 \beta \sin(\omega\tau)$$
$$\alpha_1 - 1 = \alpha_1 \beta \cos(\omega\tau)$$

Hence,

$$\omega = \sqrt{\alpha_1^2 \beta^2 - (\alpha_1 - 1)^2} \tag{11.6}$$

For real frequencies, the coefficient under the radical must be positive; that is, $\alpha_1^2\beta^2 - (\alpha_1 - 1)^2 > 0$. In what follows, we take β to be the bifurcation parameter and denote its critical value that separates stable from unstable trivial solutions by β_c and the corresponding value of ω by ω_c.

It follows from (11.5) that

$$\text{Real}\left(\frac{d(\sigma + i\omega)}{d\beta}\right) = \frac{-\alpha_1[\cos(\omega_c\tau) - \alpha_1\beta\tau]}{[\cos(\omega_c\tau) - \alpha_1\beta\tau]^2 + \sin^2(\omega_c\tau)} \neq 0$$

at $(\beta, \omega) = (\beta_c, \omega_c)$ and hence the bifurcation is a Hopf bifurcation. Next, we construct the normal form of this bifurcation by using the method of multiple scales in the next section and center-manifold reduction in Section 11.1.2.

11.1.1
The Method of Multiple Scales

We seek a uniform second-order approximate solution of (11.2) in the form

$$x(t;\epsilon) = \epsilon x_1(T_0, T_2) + \epsilon^2 x_2(T_0, T_2) + \epsilon^3 x_3(T_0, T_2) + \cdots \tag{11.7}$$

where $T_0 = t$, $T_2 = \epsilon^2 t$, and ϵ is a nondimensional bookkeeping parameter. The solution does not depend on the slow scale $T_1 = \epsilon t$ because secular terms first appear at $O(\epsilon^3)$. The derivative with respect to t is transformed into

$$\frac{d}{dt} = \frac{\partial}{dT_0} + \epsilon^2\frac{\partial}{\partial T_2} + \cdots = D_0 + \epsilon^2 D_2 + \cdots \tag{11.8}$$

Moreover, we express $x(t-\tau)$ in terms of the scales T_0 and T_2 as

$$x(t-\tau;\epsilon) = \sum_{m=1}^{3} \epsilon^m x_m\left(T_0 - \tau, T_2 - \epsilon^2\tau\right) + \cdots$$

which upon expansion for small ϵ becomes

$$x(t-\tau;\epsilon) = \sum_{m=1}^{3} \epsilon^m x_m(T_0 - \tau, T_2) - \epsilon^3\tau D_2 x_1(T_0 - \tau, T_2) + \cdots \tag{11.9}$$

Next, we introduce the detuning parameter δ to describe the nearness of β to the critical value β_c defined by

$$\beta = \beta_c + \epsilon^2\delta$$

Substituting (11.7)–(11.9) into (11.2) and equating coefficients of like powers of ϵ, we obtain

$$D_0x_1 + (1 - \alpha_1)x_1 + \alpha_1\beta_c x_{1\tau} = 0 \tag{11.10}$$

$$D_0x_2 + (1 - \alpha_1)x_2 + \alpha_1\beta_c x_{2\tau} = -\alpha_2\left(x_1 - \beta_c x_{1\tau}\right)^2 \tag{11.11}$$

$$D_0 x_3 + (1 - \alpha_1) x_3 + \alpha_1 \beta_c x_{3\tau} = -D_2 x_1 - \alpha_1 \delta x_{1\tau} + \alpha_1 \beta_c \tau D_2 x_{1\tau} - 2\alpha_2 (x_1 - \beta_c x_{1\tau})(x_2 - \beta_c x_{2\tau}) - \alpha_3 (x_1 - \beta_c x_{1\tau})^3 \quad (11.12)$$

where $x_i = x_i(T_0, T_2)$ and $x_{i\tau} = x_i(T_0 - \tau, T_2)$.

The general solution of (11.10) can be expressed as

$$x_1 = A(T_2) e^{i\omega_c T_0} + \bar{A}(T_2) e^{-i\omega_c T_0} + \sum_{m=1}^{\infty} \left[A_m(T_2) e^{(\sigma_m + i\omega_m) T_0} + \bar{A}_m(T_2) e^{(\sigma_m - i\omega_m) T_0} \right] \quad (11.13)$$

where ω_c is the critical frequency corresponding to $\sigma = 0$ on the stability boundary and it is given by (11.6); the $\sigma \pm i\omega_m$ are the remaining roots of (11.5); and the function $A(T_2)$ is determined by eliminating the secular terms at $O(\epsilon^3)$. Near the stability boundary, all of the eigenvalues have negative real parts except the eigenvalue corresponding to ω_c whose real part changes sign as the stability boundary is crossed. Hence, as time increases all of the terms under the summation in (11.13) decay with time, leaving only the first two terms. Therefore, the long-time behavior of the system is given by

$$x_1 = A(T_2) e^{i\omega_c T_0} + \bar{A}(T_2) e^{-i\omega_c T_0} \quad (11.14)$$

Substituting (11.14) into (11.11) yields

$$D_0 x_2 + (1 - \alpha_1) x_2 + \alpha_1 \beta_c x_{2\tau} = -\alpha_2 \gamma^2 A^2 e^{2i\omega_c T_0} - \alpha_2 \gamma \bar{\gamma} A \bar{A} + \text{cc} \quad (11.15)$$

whose solution can be expressed as

$$x_2 = -\alpha_2 \gamma^2 \Gamma e^{2i\omega_c T_0} - \alpha_2 \gamma \bar{\gamma} \Gamma_0 A \bar{A} + \text{cc} \quad (11.16)$$

and

$$\gamma = \beta_c e^{-i\omega_c \tau} - 1, \quad \Gamma = \frac{1}{2i\omega_c + 1 - \alpha_1 + \alpha_1 \beta_c e^{-2i\omega_c \tau}}, \quad \Gamma_0 = \frac{1}{1 - \alpha_1 + \alpha_1 \beta_c} \quad (11.17)$$

Substituting (11.14) and (11.16) into (11.12) and eliminating the terms that lead to secular terms, we obtain the complex-valued form of the normal form of the bifurcation

$$A' = -\frac{\alpha_1 \delta e^{-i\omega_c \tau}}{1 - \alpha_1 \beta_c \tau e^{-i\omega_c \tau}} A + \frac{\gamma^2 \bar{\gamma} \left[3\alpha_3 - 2\alpha_2^2 \left(\Gamma \left(1 - e^{-2\omega_c \tau} \right) + 2(1 - \beta_c) \Gamma_0 \right) \right]}{1 - \alpha_1 \beta_c \tau e^{-i\omega_c \tau}} A^2 \bar{A} \quad (11.18)$$

11.1.2
Center-Manifold Reduction

We start by converting (11.2) into an operator differential equation. We note that, whereas the solution of (11.2) when $\tau = 0$ depends on a single value, the solution of (11.2) with $\tau \neq 0$ depends on an entire set of values of x in the interval $-\tau \leq t \leq 0$. Hence, the initial space is a function space. Consequently, the delay differential equation 11.2 can be expressed as an abstract evolution equation (Campbell *et al.*, 1995; Hale, 1977; Kuang, 1993) on the Banach space $\mathcal{B}$ of continuously differentiable complex functions from $[-\tau, 0]$ to C^2 of complex functions with the norm $||p|| = \max_{-\tau\leq\theta\leq 0} |p(\theta)|$; that is,

$$\dot{x}_t = \mathcal{A}x_t + F(x_t) \tag{11.19}$$

where $x_t(\theta)\epsilon\mathcal{B}$ is defined by the shift operator

$$x_t(\theta) = x(t+\theta) \quad \text{for} \quad -\tau \leq \theta \leq 0 \tag{11.20}$$

The linear operator $\mathcal{A}$ is defined by

$$\mathcal{A}p(\theta) = \begin{cases} \dfrac{d}{d\theta}p(\theta) & \text{for} \quad -\tau \leq \theta \leq 0 \\ (\alpha_1 - 1)p(0) - \alpha_1\beta_c p(-\tau) & \text{for} \quad \theta = 0 \end{cases} \tag{11.21}$$

and the operator F can be written as

$$F = \begin{cases} 0 & \text{for} \quad -\tau \leq \theta \leq 0 \\ f & \text{for} \quad \theta = 0 \end{cases} \tag{11.22}$$

where

$$f = -\epsilon^2\alpha_1\delta x(t-\tau) - \alpha_2\left[x(t) - \beta_c x(t-\tau)\right]^2 - \alpha_3\left[x(t) - \beta_c x(t-\tau)\right]^3 \tag{11.23}$$

To carry out the analysis, we need an inner product and the adjoint operator associated with (11.21). In contrast with C^2, the space $\mathcal{B}$ does not have a natural inner product associated with its norm. However, following Hale (1977), we introduce the following bilinear form that acts like an "inner product":

$$\langle q, p\rangle = \bar{q}(0)p(0) + \int_{-\tau}^{0} \bar{q}(\xi+\tau)(-\alpha_1\beta_c)p(\xi)d\xi \tag{11.24}$$

where $q \in \mathcal{B}^*$, the space of continuously differentiable functions from $[0, \tau]$ to C^2 with the norm $||q|| = \max_{0\leq\theta\leq\tau} |q(\theta)|$. With this bilinear form, one can construct the following formal adjoint operator (Hale, 1977) associated with (11.21):

$$\bar{\mathcal{A}}q(\theta) = \begin{cases} -\dfrac{d}{d\theta}q(\theta) & \text{for} \quad 0 \leq \theta \leq \tau \\ (\alpha_1 - 1)q(0) - \alpha_1\beta_c q(\tau) & \text{for} \quad \theta = 0 \end{cases} \tag{11.25}$$

The linear operator $\mathcal{A}$ has an infinite number of eigenvalues λ_i corresponding to an infinite number of eigenfunctions p_i. At the critical value β_c, all of the eigenvalues lie in the left-half of the complex plane except the eigenvalues $\pm i\omega_c$. Therefore, in the neighborhood of β_c, the infinite-dimensional phase space of the solutions of the linear part of (11.19) can be split into a two-dimensional center subspace, corresponding to the eigenvalues $\pm i\omega_c$, and an infinite-dimensional stable subspace, corresponding to the eigenvalues with negative real parts. Consequently, the nonlinear system has a two-dimensional attractive subsystem (the center manifold) and the solutions on this manifold determine the long-time behavior of the full system.

The center subspace of $\mathcal{A}$ is spanned by the function satisfying the following boundary-value problem:

$$\frac{dp}{d\theta}(\theta) = i\omega_c p(\theta) \quad \text{for} \quad -\tau \le \theta \le 0$$
$$(\alpha_1 - 1)p(0) - \alpha_1\beta_c p(-\tau) = i\omega_c p(0)$$

whose solution can be expressed as

$$p(\theta) = e^{i\omega_c\theta} \tag{11.26}$$

on account of (11.6). The center-manifold reduction requires calculation of the adjoint at the critical value β_c. The eigenvalue corresponding to the adjoint is $-i\omega_c$. Therefore, it follows from (11.25) that

$$\frac{dq}{d\theta}(\theta) = i\omega_c q(\theta)$$
$$(\alpha_1 - 1)q(0) - \alpha_1\beta_c q(\tau) = -i\omega_c q(0)$$

whose general solution can be expressed as

$$q(\theta) = be^{i\omega_c\theta} \tag{11.27}$$

where b is a constant. To determine this constant, we impose the condition $\langle q, p\rangle = 1$, where the "inner product" is defined in (11.24). Consequently,

$$\bar{b} = \frac{1}{1 - \alpha_1\beta_c\tau e^{-i\omega_c\tau}} \tag{11.28}$$

Having determined the center subspace and its adjoint, we decompose $x_t(\theta)$ into two components: $y(t)p(\theta) + \bar{y}(t)\bar{p}(\theta)$ lying in the center subspace and the infinite-dimensional component $u_t(\theta)$ transverse to the center subspace; that is,

$$x_t(\theta) = y(t)p(\theta) + \bar{y}(t)\bar{p}(\theta) + u_t(\theta) \tag{11.29}$$

where $\langle p, u\rangle = 0$ and $\langle \bar{p}, u\rangle = 0$. Substituting (11.29) into (11.19) yields

$$\dot{y}(t)p(\theta) + \dot{\bar{y}}(t)\bar{p}(\theta) + \dot{u}_t(\theta) = i\omega_c p(\theta)y - i\omega_c\bar{p}(\theta)\bar{y} + \mathcal{A}u_t(\theta) + F(py + \bar{p}\bar{y} + u_t) \tag{11.30}$$

Taking the "inner product" of (11.30) with q, we obtain

$$\begin{aligned}\dot{\gamma} &= i\omega_c\gamma - \bar{b}\alpha_1\delta\left(\gamma e^{-i\omega_c\tau} + \bar{\gamma} e^{-i\omega_c\tau}\right) - \bar{b}\alpha_2\left(\gamma\,y + \bar{\gamma}\,\bar{y}\right)^2 \\ &\quad + \bar{b}\alpha_3\left(\gamma\,y + \bar{\gamma}\,\bar{y}\right)^3 + 2\bar{b}\alpha_2\left(\gamma\,y + \bar{\gamma}\,\bar{y}\right)\left[u(t,0) - \beta_c u(t,-\tau)\right] + \text{HOT}\end{aligned} \tag{11.31}$$

where HOT stands for higher-order terms.

Substituting (11.31) into (11.30), we obtain

$$\dot{u}_t(\theta) = \mathcal{A}u_t(\theta) - \alpha_2\left(\gamma\,y + \bar{\gamma}\,\bar{y}\right)^2\left[\bar{b}p(\theta) + b\bar{p}(\theta)\right] + \hat{F}(p\,y + \bar{p}\,\bar{y}) + \text{HOT} \tag{11.32}$$

where

$$\hat{F} = \begin{cases} 0 & \text{for} \quad -\tau \le \theta \le 0 \\ \alpha_2\left(\gamma\,y + \bar{\gamma}\,\bar{y}\right)^2\left[\bar{b}p(0) + b\bar{p}(0)\right] & \text{for} \quad \theta = 0 \\ -\alpha_2\left(\gamma\,y + \bar{\gamma}\,\bar{y}\right)^2 & \end{cases} \tag{11.33}$$

We seek an approximate solution of (11.32) in the form

$$u_t(\theta) = \alpha_2\gamma^2 h_2(\theta)y^2(t) + \alpha_2\gamma\bar{\gamma}h_0(\theta)y(t)\bar{y}(t) + \alpha_2\bar{\gamma}^2\bar{h}_2(\theta)\bar{y}^2(t) \tag{11.34}$$

Substituting (11.34) into (11.32), using (11.33), and separating the coefficients of y^2, $y\bar{y}$, and $\bar{y}^2$, we obtain

$$\mathcal{A}h_2 = 2i\omega_c h_2 - \bar{b}p - b\bar{p} \quad \text{for} \quad -\tau \le \theta \le 0 \tag{11.35}$$

$$\mathcal{A}h_2 = 2i\omega_c h_2 - \bar{b}p - b\bar{p} + 1 \quad \text{for} \quad \theta = 0 \tag{11.36}$$

$$\mathcal{A}h_0 = -2\bar{b}p - 2b\bar{p} \quad \text{for} \quad -\tau \le \theta \le 0 \tag{11.37}$$

$$\mathcal{A}h_0 = -2\bar{b}p - 2b\bar{p} + 2 \quad \text{for} \quad \theta = 0 \tag{11.38}$$

The solution of (11.35) and (11.36) can be expressed as

$$h_2 = -\frac{i\bar{b}}{\omega_c}p - \frac{ib}{3\omega_c}\bar{p} - \Gamma e^{2i\omega_c\theta} \tag{11.39}$$

where Γ is defined in (11.17). Similarly, the solution of (11.37) and (11.38) can be expressed as

$$h_0 = \frac{2i\bar{b}}{\omega_c}p - \frac{2ib}{\omega_c}\bar{p} - 2\Gamma_0 \tag{11.40}$$

where Γ_0 is defined in (11.17). Combining (11.34), (11.39), and (11.40) yields

$$\begin{aligned}u_t(\theta) &= -\alpha_2\gamma^2 y^2\left(\frac{i\bar{b}}{\omega_c}p + \frac{ib}{3\omega_c}\bar{p} - \Gamma e^{2i\omega_c\theta}\right) \\ &\quad - \alpha_2\gamma\bar{\gamma}\,y\,\bar{y}\left(-\frac{i\bar{b}}{\omega_c}p + \frac{ib}{\omega_c}\bar{p} - \Gamma_0\right) + \text{cc}\end{aligned} \tag{11.41}$$

Substituting (11.41) into (11.31) and neglecting the nonresonance terms, we obtain

$$\dot{y} = i\omega_c y - \bar{b}\alpha_1\delta y e^{-i\omega_c\tau} - \bar{b}\alpha_2 (\gamma y + \bar{\gamma}\bar{y})^2 + 3\bar{b}\alpha_3\gamma^2\bar{\gamma}y^2\bar{y}$$
$$- 2\bar{b}\alpha_2^2\gamma^2\bar{\gamma}\left[\frac{7ib\bar{\gamma}}{3\omega_c} - \frac{i\bar{b}\gamma}{\omega_c} + \Gamma\left(1 - e^{-2i\omega_c\tau}\right) + 2(1-\beta_c)\Gamma_0\right] + \text{NRT} \tag{11.42}$$

After some algebraic manipulations and changing some notations, one can show that (11.42) is in full agreement with the result obtained by Liao, Wong, and Wu (2001).

To determine the normal form of the bifurcation, we introduce the near-identify transformation

$$y = z + k_{11}z^2 + k_{12}z\bar{z} + k_{13}\bar{z}^2 \tag{11.43}$$

into (11.42) and choose the *k*s to eliminate the quadratic terms. Substituting (11.43) into (11.42) yields

$$\dot{z} + 2k_{11}z\dot{z} + k_{12}z\dot{\bar{z}} + k_{12}\dot{z}\bar{z} + 2k_{13}\dot{\bar{z}}\dot{z}$$
$$= i\omega_c\left(z + k_{11}z^2 + k_{12}z\bar{z} + k_{13}\bar{z}^2\right)$$
$$+ \bar{b}\delta\alpha_1 e^{-i\omega_c\tau}z - 2\bar{b}\alpha_2\left(\gamma^2 k_{12} + \gamma\bar{\gamma}\bar{k}_{12} + \gamma\bar{\gamma}k_{11} + \bar{\gamma}\bar{k}_{13}\right)z^2\bar{z}$$
$$- \bar{b}\alpha_2(\gamma z + \bar{\gamma}\bar{z})^2 - 2\bar{b}\alpha_2^2\gamma^2\bar{\gamma}\left[\frac{7ib\bar{\gamma}}{3\omega_c} - \frac{i\bar{b}\gamma}{\omega_c} + \Gamma\left(1 - e^{-2i\omega_c\tau}\right)\right]$$
$$+ 3\bar{b}\alpha_3\gamma^2\bar{\gamma}z^2\bar{z} + \text{NRT} \tag{11.44}$$

It follows from (11.44) that $\dot{z} \approx i\omega_c z$ and hence

$$\dot{z} = i\omega_c z + \bar{b}\delta\alpha_1 e^{-i\omega_c\tau}z - \left(i\omega_c k_{11} + \bar{b}\alpha_2\gamma^2\right)z^2$$
$$+ \left(i\omega_c k_{12} - 2\bar{b}\alpha_2\gamma\bar{\gamma}\right)z\bar{z} + \left(3i\omega_c k_{13} - \bar{b}\alpha_2\right)\bar{z}^2$$
$$- 2\bar{b}\alpha_2\left(\gamma^2 k_{12} + \gamma\bar{\gamma}\bar{k}_{12} + \gamma\bar{\gamma}k_{11} + \bar{\gamma}^2\bar{k}_{13}\right)z^2\bar{z}$$
$$- 2\bar{b}\alpha_2^2\gamma^2\bar{\gamma}\left[\frac{7ib\bar{\gamma}}{3\omega_c} - \frac{i\bar{b}\gamma}{\omega_c} + \Gamma\left(1 - e^{-2i\omega_c\tau}\right) + 2(1-\beta_c)\Gamma_0\right]z^2\bar{z}$$
$$+ 3\bar{b}\alpha_3\gamma^2\bar{\gamma}z^2\bar{z} + \text{NRT} \tag{11.45}$$

Choosing the ks to eliminate z^2, $z\bar{z}$, and $\bar{z}^2$, we obtain

$$k_{11} = \frac{i\bar{b}\alpha_2\gamma^2}{\omega_c} \tag{11.46}$$

$$k_{12} = -\frac{2i\bar{b}\alpha_2\gamma\bar{\gamma}}{\omega_c} \tag{11.47}$$

$$k_{13} = -\frac{i\bar{b}\alpha_2\bar{\gamma}^2}{3\omega_c} \tag{11.48}$$

Substituting (11.46)–(11.48) into (11.45) yields the following complex-valued normal form of the Hopf bifurcation:

$$\begin{aligned}\dot{z} &= i\omega_c z + \bar{b}\delta\alpha_1 e^{-i\omega_c\tau} z \\ &\quad + \bar{b}\gamma^2\bar{\gamma}\left[3\alpha_3 - 2\alpha_2^2\left(\left(1 - e^{-2i\omega_c\tau}\right) + 2(1-\beta_c)\Gamma_0\right)\right] z^2\bar{z}\end{aligned} \tag{11.49}$$

Letting

$$z = e^{i\omega_c t} A(t)$$

in (11.49) and using (11.28), we obtain (11.18) obtained by using the method of multiple scales.

11.2 A Single-Degree-of-Freedom System

In this section, we consider a retarded one-degree-of-freedom nonlinear system, namely the regenerative model of chatter in a lathe machine tool. The nondimensional form of the governing equation can be written as (Hanna and Tobias, 1974)

$$\ddot{x} + 2\zeta\dot{x} + x + w\left(x - x_\tau\right) + w\alpha_2\left(x - x_\tau\right)^2 + w\alpha_3\left(x - x_\tau\right)^3 = 0 \tag{11.50}$$

where $x(t)$ is the dynamic displacement of the tool tip, $x_\tau = x(t - \tau)$, τ is the period of the spindle rotation, ζ is the damping ratio, w is the width of cut, and α_2 and α_3 are coefficients that characterize the nonlinear part of the cutting force. The normal form of the Hopf bifurcation of (11.50) was constructed by using the method of multiple scales by Nayfeh, Chin, and Pratt (1997) and Nayfeh (2006) and by using a combination of the center-manifold theorem and the method of normal forms by Kalmár-Nagy, Stépán, and Moon (2001) and Gilsinn (2002).

Clearly, the trivial solution $x(t) = 0$ is a solution of (11.50). The undamped linearized form of (11.50) is

$$\ddot{x} + 2\zeta\dot{x} + x + w\left(x - x_\tau\right) = 0 \tag{11.51}$$

It has solutions of the form

$$x = x_0 e^{(\sigma + i\omega)t} \tag{11.52}$$

where σ is the growth or decay rate, ω is the frequency of oscillations, and x_0 depends on the initial conditions. Substituting (11.52) into (11.51) yields

$$(\omega - i\sigma)^2 - 2i(\omega - i\sigma)\zeta - 1 + w\left(e^{-\sigma\tau - i\omega\tau} - 1\right) = 0 \tag{11.53}$$

Separating (11.53) into real and imaginary parts yields the characteristic equations

$$1 + w + 2\zeta\sigma + \sigma^2 - \omega^2 - we^{-\sigma\tau}\cos(\omega\tau) = 0 \tag{11.54}$$

$$2(\zeta + \sigma)\omega + e^{-\sigma\tau}w\sin(\omega\tau) = 0 \tag{11.55}$$

For a given w and time delay τ, (11.54) and (11.55) have infinitely many values of σ and ω.

When $\sigma > 0$ the system response grows exponentially with time and it is unstable; when $\sigma < 0$ the system response decays exponentially with time and it is stable; and $\sigma = 0$ defines the stability boundary. To locate this boundary, we let $\sigma = 0$ in (11.53)–(11.55) and obtain the complex-valued characteristic equation

$$\omega^2 - 2i\omega\zeta - 1 + w\left(e^{-i\omega\tau} - 1\right) = 0$$

and the real-valued equations

$$1 + w - \omega^2 - w\cos(\omega\tau) = 0 \tag{11.56}$$

$$2\zeta\omega + w\sin(\omega\tau) = 0 \tag{11.57}$$

The value $\sigma = 0$ corresponds to a bifurcation separating stable from unstable trivial solutions. This bifurcation $(\sigma, w, \omega) = (0, w_c, \omega_c)$ is a Hopf bifurcation because it results from two complex-conjugate eigenvalues $\sigma + i\omega$ transversely crossing the imaginary axis. To show this, we differentiate (11.53) with respect to $\sigma + i\omega$ and w, evaluate the result at $(\sigma, w, \omega) = (0, w_c, \omega_c)$, and obtain

$$\frac{d(\sigma + i\omega)}{dw} = \frac{1}{2i\omega_c + 2i\zeta + i\omega_c w_c \tau e^{-i\omega_c\tau}} \tag{11.58}$$

whose real part is different from zero.

To determine the normal form of the Hopf bifurcation, we let

$$w = w_c + \epsilon^2\delta \tag{11.59}$$

where ϵ is a nondimensional parameter that describes the nearness of w to the critical value w_c. Next, we construct the normal form at $(w, \omega, x) = (w_c, \omega_c, 0)$ using the method of multiple scales in the next section and center-manifold reduction in Section 11.2.2.

11.2.1 The Method of Multiple Scales

We seek a second-order uniform expansion of the solution of (11.50) in the neighborhood of $w = w_c$ in the form

$$x(t;\epsilon) = \epsilon x_1(T_0, T_2) + \epsilon^2 x_2(T_0, T_2) + \epsilon^3 x_3(T_0, T_2) + \cdots \tag{11.60}$$

Again, the solution does not depend on the slow scale $T_1 = \epsilon t$ because secular terms first appear at $O(\epsilon^3)$. Moreover, we express x_τ in terms of the scales $T_0 = t$ and $T_2 = \epsilon^2 t$ as

$$\begin{aligned} x_\tau &= \epsilon x_1 \left(T_0 - \tau, T_2 - \epsilon^2 \tau\right) + \epsilon^2 x_2 \left(T_0 - \tau, T_2 - \epsilon^2 \tau\right) \\ &\quad + \epsilon^3 x_3 \left(T_0 - \tau, T_2 - \epsilon^2 \tau\right) + \cdots \end{aligned}$$

which upon expansion for small ϵ becomes

$$\begin{aligned} x_\tau &= \epsilon x_1 \left(T_0 - \tau, T_2\right) + \epsilon^3 x_3 \left(T_0 - \tau, T_2\right) + \epsilon^2 x_2 \left(T_0 - \tau, T_2\right) \\ &\quad - \epsilon^3 \tau D_2 x_1 \left(T_0 - \tau, T_2\right) + \cdots \end{aligned} \tag{11.61}$$

Substituting (11.59)–(11.61) into (11.50) and equating coefficients of like powers of ϵ, we obtain

$$D_0^2 x_1 + 2\zeta D_0 x_1 + x_1 + w_c \left(x_1 - x_{1\tau}\right) = 0 \tag{11.62}$$

$$D_0^2 x_2 + 2\zeta D_0 x_2 + x_2 + w_c \left(x_2 - x_{2\tau}\right) = -w_c \alpha_2 \left(x_1 - x_{1\tau}\right)^2 \tag{11.63}$$

$$\begin{aligned} &D_0^2 x_3 + 2\zeta D_0 x_3 + x_3 + w_c \left(x_3 - x_{3\tau}\right) \\ &= -2 D_0 D_2 x_1 - 2\zeta D_2 x_1 - \delta \left(x_1 - x_{1\tau}\right) \\ &\quad - w_c \tau D_2 x_{1\tau} - 2\alpha_2 w_c \left(x_1 - x_{1\tau}\right)\left(x_2 - x_{2\tau}\right) - w_c \alpha_3 \left(x_1 - x_{1\tau}\right)^3 \end{aligned} \tag{11.64}$$

where $x_i = x_i(T_0, T_2)$ and $x_{i\tau} = x_i(T_0 - \tau, T_2)$.

The general solution of (11.62) can be expressed as

$$\begin{aligned} x_1 &= A(T_2) e^{i\omega_c T_0} + \bar{A}(T_2) e^{-i\omega_c T_0} + \sum_{m=1}^{\infty} \left[A_m(T_2) e^{(\sigma_m + i\omega_m) T_0} \right. \\ &\quad \left. + \bar{A}_m(T_2) e^{(\sigma_m - i\omega_m) T_0} \right] \end{aligned} \tag{11.65}$$

where ω_c is the critical frequency corresponding to $\sigma = 0$ on the stability boundary and it is given by (11.56) and (11.57); the $\sigma \pm i\omega_m$ are the remaining roots of (11.56) and (11.57); and the function $A(T_2)$ is determined by eliminating the secular terms at $O(\epsilon^3)$. Near the stability boundary, all of the eigenvalues have negative real parts except the eigenvalues corresponding to $\pm i\omega_c$ whose real part changes sign as the stability boundary is crossed. Hence, as time increases, all of the terms in (11.65) decay with time, leaving only the first two terms. Therefore, the long-time behavior of the system is given by

$$x_1 = A(T_2) e^{i\omega_c T_0} + \bar{A}(T_2) e^{-i\omega_c T_0} \tag{11.66}$$

Substituting (11.66) into (11.63) yields

$$D_0^2 x_2 + 2\zeta D_0 x_2 + x_2 + w_c \left(x_2 - x_{2\tau}\right) = -w_c \alpha_c \left[\gamma^2 A^2 e^{2i\omega_c T_0} + \gamma \bar{\gamma} A \bar{A}\right] + \text{cc} \tag{11.67}$$

where cc stands for the complex conjugate of the preceding terms and

$$\gamma = e^{-iw_c\tau} - 1 \tag{11.68}$$

A particular solution of (11.67) that does not include the homogeneous solution can be expressed as

$$x_2 = w_c\alpha_2\gamma^2\Gamma A^2 e^{2i\omega_c T_0} - \omega_c\alpha_2\gamma\bar{\gamma}A\bar{A} + \text{cc} \tag{11.69}$$

where

$$\Gamma = \frac{1}{4\omega_c^2 - 1 - 4i\omega\zeta + w_c\left(e^{-2i\omega_c\tau} - 1\right)} \tag{11.70}$$

Substituting (11.66) and (11.69) into (11.64), we obtain

$$D_0^2x_3 + 2\zeta D_0x_3 + x_3 + w_c(x_3 - x_{3\tau}) = -\left[\left(2i\omega_c + 2\zeta + \omega_c\tau e^{-i\omega_c\tau}\right)A' - \delta\gamma A - \Lambda A^2\bar{A}\right]e^{i\omega_c T_0} + \text{NST} + \text{cc} \tag{11.71}$$

where

$$\Lambda = 3w_c\alpha_3\gamma^2\bar{\gamma} + 2w_c^2\alpha_2^2\gamma^2\bar{\gamma}\left(1 - e^{-2i\omega_c\tau}\right)\Gamma \tag{11.72}$$

Eliminating the terms that produce secular terms in (11.71) yields the following complex-valued normal form of the Hopf bifurcation:

$$\left(2i\omega_c + 2\zeta + w_c\tau e^{-i\omega_c\tau}\right)A' = \delta\gamma A + \Lambda A^2\bar{A} \tag{11.73}$$

Next, we introduce the polar form

$$A = \tfrac{1}{2}ae^{i\beta} \tag{11.74}$$

into (11.73), separate real and imaginary parts, and obtain the following real-valued normal form of the bifurcation:

$$a' = \delta\chi_1 a + \chi_3 a^3 \tag{11.75}$$

$$\beta' = \delta\chi_2 + \chi_4 a^2 \tag{11.76}$$

where

$$\chi_1 = \frac{(w_c\tau\cos\omega_c\tau + 2\zeta)(\cos\omega_c\tau - 1) - (2\omega_c - w_c\tau\sin\omega_c\tau)\sin\omega_c\tau}{D} \tag{11.77}$$

$$\chi_2 = -\frac{(w_c\tau\cos\omega_c\tau + 2\zeta)\sin\omega_c\tau + (2\omega_c - w_c\tau\sin\omega_c\tau)(\cos\omega_c\tau - 1)}{D} \tag{11.78}$$

$$\chi_3 = -\frac{\Lambda_r(w_c\tau\cos\omega_c\tau + 2\zeta) + \Lambda_i(2\omega_c - w_c\tau\sin\omega_c\tau)}{4D} \tag{11.79}$$

$$\chi_4 = -\frac{\Lambda_i(w_c\tau\cos\omega_c\tau + 2\zeta) - \Lambda_r(2\omega_c - w_c\tau\sin\omega_c\tau)}{4D} \tag{11.80}$$

$$D = (w_c\tau\cos\omega_c\tau + 2\zeta)^2 + (2\omega_c - w_c\tau\sin\omega_c\tau)^2 \tag{11.81}$$

11.2.2
Center-Manifold Reduction

We start by writing (11.50) in the vector form

$$\dot{\boldsymbol{x}} = L\boldsymbol{x}(t) + R\boldsymbol{x}(t-\tau) + \hat{\boldsymbol{f}}\left[\boldsymbol{x}(t), \boldsymbol{x}(t-\tau)\right] \tag{11.82}$$

where

$$L = \begin{bmatrix} 0 & 1 \\ -1-w_c & -2\zeta \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 \\ w_c & 0 \end{bmatrix} \tag{11.83}$$

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \hat{\boldsymbol{f}} = \begin{bmatrix} 0 \\ f \end{bmatrix} \tag{11.84}$$

$$f = \epsilon^2\delta\left[x_1(t) - x_1(t-\tau)\right] - w_c\alpha_2\left[x_1(t) - x_1(t-\tau)\right]^2 - w_c\alpha_3\left[(x_1(t) - x_1(t-\tau)\right]^3 \tag{11.85}$$

Then, we represent (11.82) as an operator differential equation. The result is Gilsinn (2002), Kalmár-Nagy, Stépán, and Moon (2001)

$$\dot{\boldsymbol{x}}_t = \mathcal{A}\boldsymbol{x}_t + \boldsymbol{F}(\boldsymbol{x}_t) \tag{11.86}$$

where $\boldsymbol{x}_t(\theta)\epsilon\mathcal{B}$ is defined by the shift operator

$$\boldsymbol{x}_t(\theta) = \boldsymbol{x}(t+\theta) \quad \text{for} \quad -\tau \le \theta \le 0 \tag{11.87}$$

and the linear operator $\mathcal{A}$ is defined by

$$\mathcal{A}\boldsymbol{p}(\theta) = \begin{cases} \dfrac{d}{d\theta}\boldsymbol{p}(\theta) & \text{for} \quad -\tau \le \theta \le 0 \\ L\boldsymbol{p}(0) + R\boldsymbol{p}(-\tau) & \text{for} \quad \theta = 0 \end{cases} \tag{11.88}$$

Moreover, the operator $\boldsymbol{F}$ can be written as

$$\boldsymbol{F} = \begin{cases} 0 & \text{for} \quad -\tau \le \theta \le 0 \\ \hat{\boldsymbol{f}} & \text{for} \quad \theta = 0 \end{cases} \tag{11.89}$$

Again, to carry out the analysis, we need the adjoint operator associated with (11.88). It can be written as

$$\mathcal{A}^*\boldsymbol{q}(\theta) = \begin{cases} -\dfrac{d}{d\theta}\boldsymbol{q}(\theta) & \text{for} \quad 0 \le \theta \le \tau \\ L^*\boldsymbol{q}(0) + R^*\boldsymbol{q}(\tau) & \text{for} \quad \theta = 0 \end{cases} \tag{11.90}$$

where the superscript * stands for the complex conjugate of the transpose. The "inner product" in this formulation assumes the form

$$\langle \boldsymbol{q}, \boldsymbol{p} \rangle = \boldsymbol{q}^*(0)\boldsymbol{p}(0) + \int_{-\tau}^{0} \boldsymbol{q}^*\,(\xi + \tau)\, R\boldsymbol{p}(\xi)d\xi \tag{11.91}$$

Whereas Kalmár-Nagy, Stépán, and Moon (2001) carried out their analysis using real variables, Gilsinn (2002) used complex variables, in which the algebra is less involved. Moreover, to compare the results of center-manifold reduction with those obtained using the method of multiple scales, we use complex variables.

The linear operator $\mathcal{A}$ defined by (11.88) has an infinite number of eigenvalues λ_i and an infinite number of eigenfunctions $\boldsymbol{p}_i$. For small values of w, all of the eigenvalues lie in the left-half of the complex plane. As w increases, a complex-conjugate pair of these eigenvalues approaches the imaginary axis and transversely crosses it at $w = w_c$, the critical value corresponding to the Hopf bifurcation. At $w = w_c$, the infinite-dimensional phase space of the solutions of $\dot{\boldsymbol{x}}_t = \mathcal{A}\boldsymbol{x}_t$ can be split into a two-dimensional space, corresponding to the eigenvalues with zero real parts, and an infinite-dimensional stable subspace, corresponding to the eigenvalues with negative real parts. Consequently, solutions of the system given by (11.86) are locally attracted to a two-dimensional center manifold.

The center subspace of the linear operator $\mathcal{A}$ defined in (11.88) is spanned by the function satisfying the boundary-value problem

$$\frac{d}{d\theta}\boldsymbol{p}(\theta) = i\omega_c\boldsymbol{p}(\theta) \ \text{ for } \ -\tau \le \theta \le 0 \tag{11.92}$$

$$L\boldsymbol{p}(0) + R\boldsymbol{p}(-\tau) = i\omega_c\boldsymbol{p}(0) \tag{11.93}$$

The general solution of (11.92) can be written as

$$\boldsymbol{p}(\theta) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{i\omega_c\theta} \tag{11.94}$$

where c_1 and c_2 are constants. Substituting (11.94) into (11.93) yields

$$\left[L - i\omega_c I + Re^{-i\omega_c\tau}\right]\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0 \tag{11.95}$$

Because $i\omega_c$ is an eigenvalue of (11.92) and (11.93), (11.95) has nontrivial solutions. Setting $c_1 = 1$, we find that $c_2 = i\omega_c$. Hence,

$$\boldsymbol{p}(\theta) = \begin{bmatrix} 1 \\ i\omega_c \end{bmatrix} e^{i\omega_c\theta} \tag{11.96}$$

The center-manifold reduction requires calculation of the adjoint at the critical value w_c. The eigenvalue corresponding to the adjoint is $-i\omega_c$. Therefore, it follows from (11.90) that

$$\frac{d}{d\theta}\boldsymbol{q}(\theta) = i\omega_c\boldsymbol{q}(\theta)$$

whose general solution can be expressed as

$$\mathbf{q}(\theta) = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} e^{i\omega_c\theta} \tag{11.97}$$

Substituting (11.97) into (11.90) yields

$$i\omega_c b_1 + \left(w_c e^{i\omega_c\tau} - 1 - w_c\right) b_2 = 0 \tag{11.98}$$

$$b_1 + (i\omega_c - 2\zeta)\, b_2 = 0 \tag{11.99}$$

whose determinant is zero because $i\omega_c$ is an eigenvalue of the original linear problem and hence the adjoint problem. Therefore, nontrivial solutions exist for b_1 and b_2; they are not unique. To uniquely determine b_1 and b_2, we require the "inner product" $\langle \mathbf{q}, \mathbf{p} \rangle = 1$. Consequently,

$$\bar{b}_1 + i\omega_c\bar{b}_2 + \int_{-\tau}^{0} \left[\bar{b}_1\bar{b}_2\right] e^{-i\omega_c\tau} R \begin{bmatrix} 1 \\ i\omega_c \end{bmatrix} d\xi = 1$$

Hence,

$$\bar{b}_1 + \left(i\omega_c + w_c\tau e^{-i\omega_c\tau}\right) \bar{b}_2 = 1 \tag{11.100}$$

We note that $\langle \mathbf{q}, \mathbf{p}^* \rangle = 0$ or

$$\left[\bar{b}_1\bar{b}_2\right] \begin{bmatrix} 1 \\ -i\omega_c \end{bmatrix} + e^{-i\omega_c\tau} \int_{-\tau}^{0} \left[\bar{b}_1\bar{b}_2\right] \begin{bmatrix} 0 & 1 \\ w_c & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -i\omega_c \end{bmatrix} d\xi = 0$$

Hence,

$$\bar{b}_1 + \left(\frac{w_c}{\omega_c} \sin\omega_c\tau - i\omega_c\right) \bar{b}_2 = 0 \tag{11.101}$$

One can use (11.100) with any of (11.98), (11.99), and (11.101) to solve uniquely for b_1 and b_2. To be able to compare directly the results of the center-manifold reduction with the results of the method of multiple scales, we solve (11.99) and (11.100). The result is

$$\bar{b}_2 = \frac{1}{2i\omega_c + 2\zeta + w_c\tau e^{-i\omega_c\tau}} \tag{11.102}$$

$$b_1 = (2\zeta - i\omega_c)\, b_2 \tag{11.103}$$

Having determined the center subspace and its adjoint, we decompose $\mathbf{x}_t(\theta)$ into two components: $y(t)\mathbf{p}(\theta) + \bar{y}(t)\bar{\mathbf{p}}(\dot{\theta})$ lying in the center subspace and the infinite-dimensional component $\mathbf{u}_t(\theta)$ transverse to the center subspace; that is,

$$\mathbf{x}_t(\theta) = y(t)\mathbf{p}(\theta) + \bar{y}(t)\bar{\mathbf{p}}(\theta) + \mathbf{u}_t(\theta) \tag{11.104}$$

where $\langle \boldsymbol{p}, \boldsymbol{u}\rangle = 0$ and $\langle \bar{\boldsymbol{p}}, \boldsymbol{u}\rangle = 0$. Substituting (11.104) into (11.86) yields

$$\dot{\gamma}(t)\boldsymbol{p}(\theta) + \dot{\bar{\gamma}}(t)\bar{\boldsymbol{p}}(\theta) + \dot{\boldsymbol{u}}_t(\theta) = i\omega \boldsymbol{p}(\theta)\gamma - i\omega \bar{\boldsymbol{p}}(\theta)\bar{\gamma} + \mathcal{A}\boldsymbol{u}_t(\theta) + \boldsymbol{F}\left(\boldsymbol{p}\gamma + \bar{\boldsymbol{p}}\bar{\gamma} + \boldsymbol{u}_t\right) \tag{11.105}$$

Taking the "inner product" of (11.105) with $\boldsymbol{q}$, we obtain

$$\dot{\gamma} = i\omega\gamma + \langle \boldsymbol{q}, \boldsymbol{F}\rangle$$

Hence,

$$\dot{\gamma} = i\omega\gamma + \bar{b}_2\delta\left(\gamma\gamma + \bar{\gamma}\bar{\gamma}\right) - \bar{b}_2 w_c\alpha_2\left(\gamma\gamma + \bar{\gamma}\bar{\gamma}\right)^2 + \bar{b}_2 w_c\alpha_3\left(\gamma\gamma + \bar{\gamma}\bar{\gamma}\right)^3 + 2\bar{b}_2 w_c\alpha_2\left(\gamma\gamma + \bar{\gamma}\bar{\gamma}\right)\left[u_1(t,0) - u_1(t,-\tau)\right] \tag{11.106}$$

Substituting (11.106) into (11.105), we obtain

$$\dot{\boldsymbol{u}}_t(\theta) = \mathcal{A}\boldsymbol{u}_t(\theta) + w_c\alpha_2\left(\gamma\gamma + \bar{\gamma}\bar{\gamma}\right)^2\left[\bar{b}_2\boldsymbol{p}(\theta) + b_2\bar{\boldsymbol{p}}(\theta)\right] + \boldsymbol{F}(\boldsymbol{p}\gamma + \bar{\boldsymbol{p}}\bar{\gamma}) + \text{HOT} \tag{11.107}$$

where

$$F = \begin{cases} 0 & \text{for} \quad -\tau \le \theta \le 0 \\ \hat{\boldsymbol{F}} & \text{for} \quad \theta = 0 \end{cases} \tag{11.108}$$

$$\hat{\boldsymbol{F}} = w_c\alpha_2\left(\gamma\gamma + \bar{\gamma}\bar{\gamma}\right)^2\left[\bar{b}_2\boldsymbol{p}(0) + b_2\bar{\boldsymbol{p}}(0)\right] - \begin{bmatrix} 0 \\ w_c\alpha_2\left(\gamma\gamma + \bar{\gamma}\bar{\gamma}\right)^2 \end{bmatrix}$$

We seek an approximate solution of (11.107) in the form

$$\boldsymbol{u}_t(\theta) = w_c\alpha_2\gamma^2\boldsymbol{h}_2(\theta)\gamma^2(t) + w_c\alpha_2\gamma\bar{\gamma}\boldsymbol{h}_0(\theta)\gamma(t)\bar{\gamma}(t) + w_c\alpha_2\bar{\gamma}^2\bar{\boldsymbol{h}}_2(\theta)\bar{\gamma}^2(t) \tag{11.109}$$

Substituting (11.109) into (11.107), using (11.108), and separating the coefficients of γ^2, $\gamma\bar{\gamma}$, and $\bar{\gamma}^2$, we obtain

$$\mathcal{A}\boldsymbol{h}_2 = 2i\omega_c\boldsymbol{h}_2 - \left(\bar{b}_2\boldsymbol{p} + b_2\bar{\boldsymbol{p}}\right) \quad \text{for} \quad -\tau \le \theta \le 0 \tag{11.110}$$

$$\mathcal{A}\boldsymbol{h}_2 = 2i\omega_c\boldsymbol{h}_2 - \left(\bar{b}_2\boldsymbol{p} + b_2\bar{\boldsymbol{p}}\right) - w_c\alpha_2\gamma^2\begin{bmatrix}0\\1\end{bmatrix} \quad \text{for} \quad \theta = 0 \tag{11.111}$$

$$\mathcal{A}\boldsymbol{h}_0 = -2\left(\bar{b}_2\boldsymbol{p} + b_2\bar{\boldsymbol{p}}\right) \quad \text{for} \quad -\tau \le \theta \le 0 \tag{11.112}$$

$$\mathcal{A}\boldsymbol{h}_0 = -2\left(\bar{b}_2\boldsymbol{p} + b_2\bar{\boldsymbol{p}}\right) + 2\begin{bmatrix}0\\1\end{bmatrix} \quad \text{for} \quad \theta = 0 \tag{11.113}$$

The solution of (11.110) and (11.111) can be expressed as

$$\boldsymbol{h}_2 = -\frac{i\bar{b}_2}{\omega_c}\boldsymbol{p} - \frac{ib_2}{3\omega_c}\bar{\boldsymbol{p}} + \Gamma\begin{bmatrix}1\\2i\omega_c\end{bmatrix} \tag{11.114}$$

where Γ is defined in (11.70). Similarly, the solution of (11.112) and (11.113) can be expressed as

$$\boldsymbol{h}_0 = \frac{2i\bar{b}_2}{\omega_c}\boldsymbol{p} - \frac{2ib_2}{\omega_c}\bar{\boldsymbol{p}} - 2\begin{bmatrix}1\\0\end{bmatrix} \tag{11.115}$$

Combining (11.109), (11.114), and (11.115) yields

$$\begin{aligned}\boldsymbol{u}_t(\theta) &= w_c\alpha_2\gamma^2 y^2\left(-\frac{i\bar{b}_2}{\omega_c}\boldsymbol{p} - \frac{ib_2}{3\omega_c}\bar{\boldsymbol{p}} + \Gamma\begin{bmatrix}1\\2i\omega_c\end{bmatrix}e^{2i\omega_c\theta}\right)\\ &\quad + w_c\alpha_2\gamma\bar{\gamma}y\bar{y}\left(\frac{i\bar{b}_2}{\omega_c}\boldsymbol{p} - \frac{ib_2}{\omega_c}\bar{\boldsymbol{p}} - \begin{bmatrix}1\\0\end{bmatrix}\right) + \text{cc}\end{aligned} \tag{11.116}$$

Substituting (11.116) into (11.106) and neglecting the nonresonance terms, we obtain

$$\begin{aligned}\dot{y} &= i\omega_c y + \bar{b}_2\delta\gamma y - \bar{b}_2 w_c\alpha_2(\gamma y + \bar{\gamma}\bar{y})^2 + 3\bar{b}_2 w_c\alpha_3\gamma^2\bar{\gamma}y^2\bar{y}\\ &\quad + 2\bar{b}_2 w_c^2\alpha_2^2\gamma^2\bar{\gamma}\left[\frac{7ib_2\bar{\gamma}}{3\omega_c} - \frac{i\bar{b}_2\gamma}{\omega_c} + \Gamma\left(1 - e^{-2i\omega_c\tau}\right)\right] + \text{NRT}\end{aligned} \tag{11.117}$$

With some algebra and notation change, one can show that (11.117) is in full agreement with that obtained by Gilsinn (2002).

To determine the normal form of the bifurcation, we introduce the near-identity transformation

$$y = z + k_{11}z^2 + k_{12}z\bar{z} + k_{13}\bar{z}^2 \tag{11.118}$$

into (11.117) and choose the ks to eliminate the quadratic terms. Substituting (11.118) into (11.117) yields

$$\begin{aligned}&\dot{z} + 2k_{11}z\dot{z} + k_{12}z\dot{\bar{z}} + k_{12}\dot{z}\bar{z} + 2k_{13}\bar{z}\dot{\bar{z}}\\ &= i\omega_c\left(z + k_{11}z^2 + k_{12}z\bar{z} + k_{13}\bar{z}^2\right)\\ &\quad + \bar{b}_2\delta\gamma z - \bar{b}_2 w_c\alpha_2(\gamma z + \bar{\gamma}\bar{z})^2\\ &\quad - 2\bar{b}_2 w_c\alpha_2\left(\gamma^2 k_{12} + \gamma\bar{\gamma}\bar{k}_{12} + \gamma\bar{\gamma}k_{11} + \bar{\gamma}\bar{k}_{13}\right)z^2\bar{z} + 3\bar{b}_2 w_c\alpha_3\gamma^2\bar{\gamma}z^2\bar{z}\\ &\quad + 2\bar{b}_2 w_c^2\alpha_2^2\gamma^2\bar{\gamma}\left[\frac{7ib_2\bar{\gamma}}{3\omega_c} - \frac{i\bar{b}_2\gamma}{\omega_c} + \Gamma\left(1 - e^{-2i\omega_c\tau}\right)\right] + \text{NRT}\end{aligned} \tag{11.119}$$

It follows from (11.119) that $\dot{z} \approx i\omega_c z$ and hence

$$\begin{aligned}\dot{z} &= i\omega_c z + \bar{b}_2\delta\gamma z - \left(i\omega_c k_{11} + \bar{b}_2 w_c \alpha_2 \gamma^2\right) z^2 \\ &+ \left(i\omega_c k_{12} - 2\bar{b}_2 w_c \alpha_2 \gamma \bar{\gamma}\right) z\bar{z} + \left(3i\omega_c k_{13} - \bar{b}_2 w_c \alpha_2\right) \bar{z}^2 \\ &- 2\bar{b}_2 w_c \alpha_2 \left(\gamma^2 k_{12} + \gamma\bar{\gamma}\bar{k}_{12} + \gamma\bar{\gamma} k_{11} + \bar{\gamma}^2 \bar{k}_{13}\right) z^2\bar{z} \\ &+ 3\bar{b}_2 w_c \alpha_3 \gamma^2 \bar{\gamma} z^2 \bar{z} \\ &+ 2\bar{b}_2 w_c^2 \alpha_2^2 \gamma^2 \bar{\gamma} \left[\frac{7ib_2\bar{\gamma}}{3\omega_c} - \frac{i\bar{b}_2\gamma}{\omega_c} + \Gamma\left(1 - e^{-2i\omega_c\tau}\right)\right] z^2\bar{z} \\ &+ \text{NRT}\end{aligned} \tag{11.120}$$

Choosing the ks to eliminate z^2, $z\bar{z}$, and $\bar{z}^2$, we obtain

$$k_{11} = \frac{i\bar{b}_2 w_c \alpha_2 \gamma^2}{\omega_c} \tag{11.121}$$

$$k_{12} = -\frac{2i\bar{b}_2 w_c \alpha_2 \gamma\bar{\gamma}}{\omega_c} \tag{11.122}$$

$$k_{13} = -\frac{i\bar{b}_2 w_c \alpha_2 \bar{\gamma}^2}{3\omega_c} \tag{11.123}$$

Substituting (11.121)–(11.123) into (11.120) yields the following complex-valued normal form of the Hopf bifurcation:

$$\dot{z} = i\omega_c z + \bar{b}_2\delta\gamma z + \bar{b}_2\left[3w_c\alpha_3\gamma^2\bar{\gamma} + 2w_c^2\alpha_2^2\gamma^2\bar{\gamma}\Gamma\left(1 - e^{-2i\omega_c\tau}\right)\right] z^2\bar{z} \tag{11.124}$$

Letting

$$z = e^{i\omega_c\tau} A(t)$$

in (11.124) and using (11.102), we obtain (11.73) obtained by using the method of multiple scales.

11.3 A Three-Dimensional System

In this section, we consider the following three-neuron model with time delay (Babcock and Westervelt, 1987; Gopalsamy and Leung, 1996; Hopfield, 1984; Liao, Guo, and Li, 2007):

$$\dot{x}_1(t) = -x_1(t) + \alpha_1 \tanh\left[x_3(t) - \beta x_3(t-\tau)\right] \tag{11.125}$$

$$\dot{x}_2(t) = -x_2(t) + \alpha_2 \tanh\left[x_1(t) - \beta x_1(t-\tau)\right] \tag{11.126}$$

$$\dot{x}_3(t) = -x_3(t) + \alpha_3 \tanh\left[x_2(t) - \beta x_2(t-\tau)\right] \tag{11.127}$$

where the α_i, β, and τ are positive constants. Clearly, the trivial solution $x_i = 0$, $i = 1, 2, 3$ is a solution of (11.125)–(11.127). In what follows, we use β as the bifurcation parameter. For given α_i and τ, this trivial solution loses stability through a Hopf bifurcation as β exceeds a critical value β_r, as shown below. We construct the normal form of this bifurcation by using the method of multiple scalers in Section 11.3.1 and by using center-manifold reduction in Section 11.3.2. To this end, we expand (11.125)–(11.127) for small x_i, keep up to cubic terms, and obtain

$$\dot{\boldsymbol{x}} = L\boldsymbol{x}(t) - \beta R\boldsymbol{x}(t-\tau) + \boldsymbol{f}\left[\boldsymbol{x}(t), \boldsymbol{x}(t-\tau)\right] \tag{11.128}$$

where

$$L = \begin{bmatrix} -1 & 0 & \alpha_1 \\ \alpha_2 & -1 & 0 \\ 0 & \alpha_3 & -1 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 & \alpha_1 \\ \alpha_2 & 0 & 0 \\ 0 & \alpha_3 & 0 \end{bmatrix} \tag{11.129}$$

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \quad \boldsymbol{f} = \begin{bmatrix} \alpha_1\left[x_3(t) - \beta x_3(t-\tau)\right]^3 \\ \alpha_2\left[x_1(t) - \beta x_1(t-\tau)\right]^3 \\ \alpha_3\left[x_2(t) - \beta x_2(t-\tau)\right]^3 \end{bmatrix} \tag{11.130}$$

To determine the type of bifurcation of the trivial solution, we linearize (11.128) and obtain

$$\dot{\boldsymbol{x}} = L\boldsymbol{x}(t) - \beta R\boldsymbol{x}(t-\tau) \tag{11.131}$$

Seeking solutions of (11.131) in the form

$$\dot{\boldsymbol{x}} = \boldsymbol{x}_0 e^{(\sigma+i\omega)t} \tag{11.132}$$

leads to the characteristic equation

$$|L - \beta R e^{-(\sigma+i\omega)\tau} - i\omega I| = 0$$

or

$$(\sigma + 1 + i\omega)^3 - \alpha^3\left[1 - \beta e^{-(\sigma+i\omega)\tau}\right]^3 = 0$$

where $\alpha^3 = \alpha_1\alpha_2\alpha_3$. Hence, either

$$\sigma + 1 + i\omega - \alpha\left[1 - \beta e^{-(\sigma+i\omega)\tau}\right] = 0 \tag{11.133}$$

or

$$(\sigma + 1 + i\omega)^2 + \alpha(\sigma + 1 + i\omega)\left[1 - \beta e^{-(\sigma+i\omega)\tau}\right] + \alpha^2\left[1 - \beta e^{-(\sigma+i\omega)\tau}\right]^2 = 0 \tag{11.134}$$

Again, when $\sigma > 0$, the trivial solution is unstable; when $\sigma < 0$, the trivial solution is asymptotically stable; and $\sigma = 0$ separates stable from unstable trivial solutions. Putting $\sigma = 0$ in (11.133) and separating real and imaginary parts, we obtain

$$\omega = \alpha\beta\sin(\omega\tau)$$
$$\alpha - 1 = \alpha\beta\cos(\omega\tau)$$

Hence, for given α and τ, the critical value ω_c of ω is given by

$$\omega_c = (\alpha_1 - 1)\tan(\omega_c \tau) \tag{11.135}$$

Differentiating (11.133) with respect to $\sigma + i\omega$ and β and evaluating the result at the critical values yields

$$\frac{d(\sigma + i\omega)}{d\beta} = \frac{i\omega_c + 1 - \alpha}{\beta_c\,(i\omega_c + 1 - \alpha)\,\tau}$$

whose real part is different from zero. Therefore, the trivial solution loses stability as a result of a pair of complex conjugate eigenvalues transversely crossing the imaginary axis as β exceeds the critical value β_c, and hence the bifurcation is a Hopf bifurcation. Next, we construct the normal form of this bifurcation by using the method of multiple scales in the following section and by using center-manifold reduction in Section 11.3.2.

11.3.1
The Method of Multiple Scales

Because the nonlinearity is cubic, we seek a uniform second-order approximate solution of (11.128)–(11.130) in powers of $\epsilon^{1/2}$ instead of powers of ϵ as done above for cases with quadratic and cubic nonlinearities. Thus, we let

$$\boldsymbol{x}(t;\epsilon) = \epsilon^{1/2}\boldsymbol{x}_1(T_0, T_1) + \epsilon^{3/2}\boldsymbol{x}_2(T_0, T_1) + \cdots \tag{11.136}$$

where $T_0 = t$, $T_1 = \epsilon t$, and ϵ is a nondimensional bookkeeping parameter. The derivative with respect to t, in this case, is transformed into

$$\frac{d}{dt} = \frac{\partial}{dT_0} + \epsilon\frac{\partial}{\partial T_1} + \cdots = D_0 + \epsilon D_1 + \cdots \tag{11.137}$$

Moreover, we express $\boldsymbol{x}(t-\tau)$ in terms of the scales T_0 and T_1 as

$$\boldsymbol{x}(t-\tau;\epsilon) = \epsilon^{1/2}\boldsymbol{x}_1(T_0-\tau, T_1-\epsilon\tau) + \epsilon^{3/2}\boldsymbol{x}_2(T_0-\tau, T_1-\epsilon\tau) + \cdots$$

which upon expansion for small ϵ becomes

$$\begin{aligned}\boldsymbol{x}(t-\tau;\epsilon) = {} & \epsilon^{1/2}\boldsymbol{x}_1(T_0-\tau, T_1) + \epsilon^{3/2}\boldsymbol{x}_2(T_0-\tau, T_1) \\ & - \epsilon^{3/2}\tau D_1\boldsymbol{x}_1(T_0-\tau, T_1) + \cdots \end{aligned} \tag{11.138}$$

Next, we introduce the detuning parameter δ to describe the nearness of β to the critical value β_c defined by

$$\beta = \beta_c + \epsilon\delta \tag{11.139}$$

Substituting (11.136)–(11.139) into (11.128) and equating coefficients of like powers of ϵ yields

$$D_0 \boldsymbol{x}_1(T_0, T_1) - L\boldsymbol{x}_1(T_0, T_1) + \beta_c R\boldsymbol{x}_1(T_0 - \tau, T_1) = 0 \tag{11.140}$$

$$\begin{aligned} D_0 \boldsymbol{x}_2(T_0, T_1) - L\boldsymbol{x}_2(T_0, T_1) + \beta_c R\boldsymbol{x}_2(T_0 - \tau, T_1) &= -\delta R\boldsymbol{x}_1(T_0 - \tau, T_1) \\ - D_1\boldsymbol{x}_1(T_0, T_1) + \beta_c \tau R D_1 \boldsymbol{x}_1(T_0 - \tau, T_1) \\ + f\left[\boldsymbol{x}_1(T_0, T_1), \boldsymbol{x}_1(T_0 - \tau, T_1)\right] \end{aligned} \tag{11.141}$$

The nondecaying solution of (11.140) can be expressed as

$$\boldsymbol{x}_1(T_0, T_1) = A(T_1)\boldsymbol{c}e^{i\omega_c T_0} + \bar{A}(T_1)\bar{\boldsymbol{c}}e^{-i\omega_c T_0} \tag{11.142}$$

where $\boldsymbol{c}$ is given by

$$\boldsymbol{c} = \begin{bmatrix} 1 \\ c_2 \\ c_3 \end{bmatrix}, \quad c_2 = \frac{\alpha_2\left(1 - \beta_c e^{-i\omega_c \tau}\right)}{1 + i\omega_c}, \quad \text{and} \quad c_3 = \frac{1 + i\omega_c}{\alpha_1\left(1 - \beta_c e^{-i\omega_c \tau}\right)} \tag{11.143}$$

Substituting (11.142) into (11.141) yields

$$\begin{aligned} D_0 \boldsymbol{x}_2(T_0, T_1) - L\boldsymbol{x}_2(T_0, T_1) + \beta_c R\boldsymbol{x}_2(T_0 - \tau, T_1) &= -\delta R\boldsymbol{c}Ae^{-i\omega_c \tau}e^{i\omega_c T_0} \\ -\left[I - \beta_c \tau R e^{-i\omega_c \tau}\right]\boldsymbol{c}A' e^{i\omega_c T_0} - 3\gamma^2\bar{\gamma}A^2\bar{A}\hat{\boldsymbol{f}}e^{i\omega_c T_0} + \text{cc} + \text{NST} \end{aligned} \tag{11.144}$$

where

$$\gamma = \beta_c e^{-i\omega_c \tau} - 1, \quad \hat{\boldsymbol{f}} = \begin{bmatrix} \alpha_1 c_3^2 \bar{c}_3 \\ \alpha_2 \\ \alpha_3 c_2^2 \bar{c}_2 \end{bmatrix} \tag{11.145}$$

Because the homogeneous part of (11.144) has nontrivial solutions, the nonhomogeneous equation has a solution only if a solvability condition is satisfied. To determine this solvability condition, we seek a particular solution of (11.144) in the form

$$\boldsymbol{x}_2(T_0, T_1) = \phi(T_1)e^{i\omega_c T_0}$$

and obtain

$$\begin{aligned} \left[L - \beta_c R e^{-i\omega_c \tau} - i\omega_c I\right]\phi \\ = \left[I - \beta_c \tau R e^{-i\omega_c \tau}\right]\boldsymbol{c}A' + \delta R\boldsymbol{c}Ae^{-i\omega_c \tau} \\ + 3\gamma^2\bar{\gamma}\hat{\boldsymbol{f}}A^2\bar{A} \end{aligned} \tag{11.146}$$

We note that the problem of finding the solvability condition for the system of differential equations (11.144) has been transformed into finding the solvability condition of the system of algebraic equations (11.146). Again, because $i\omega_c$ is an eigenvalue of the homogeneous part of (11.146), the nonhomogeneous equation has solutions if and only if a solvability condition is satisfied. This condition demands that the right-hand side of (11.146) be orthogonal to every solution of the adjoint homogeneous problem. In this case, the adjoint is governed by

$$\left[L^* - \beta_c R^* e^{i\omega_c\tau} + i\omega_c I\right] \boldsymbol{b} = 0 \tag{11.147}$$

We note that $\boldsymbol{b}$ is not unique, and to make it unique we impose the condition

$$\boldsymbol{b}^*.\boldsymbol{c} = 1 \tag{11.148}$$

Solving (11.147) and (11.148) and using (11.145), we obtain the unique adjoint

$$\boldsymbol{b} = \tfrac{1}{3}\begin{bmatrix} 1 \\ b_2 \\ b_3 \end{bmatrix}, \quad b_2 = \frac{1 - i\omega_c}{\alpha_2(1 - \beta_c e^{i\omega_c\tau})}, \quad \text{and} \quad b_3 = \frac{\alpha_1(1 - \beta_c e^{i\omega_c\tau})}{1 - i\omega_c} \tag{11.149}$$

Taking the “inner product” of the right-hand side of (11.146) with b^* yields the solvability condition, normal form,

$$A' = \delta\Lambda_1 A + \Lambda_2 A^2 \bar{A} \tag{11.150}$$

where

$$\begin{aligned} \Lambda_1 &= -\frac{\left(\alpha_1 c_2 c_3^2 + \alpha_2 c_3 + \alpha_3 c_2^2\right) e^{-i\omega_c\tau}}{3c_2c_3 - \beta_c\tau\left(\alpha_1 c_2 c_3^2 + \alpha_2 c_3 + \alpha_3 c_2^2\right)} \\ \Lambda_2 &= -\frac{3\left(\alpha_1 c_2 c_3^3 \bar{c}_3 + \alpha_2 c_3 + \alpha_3 c_2^3 \bar{c}_2\right)\gamma^2\bar{\gamma}}{3c_2c_3 - \beta_c\tau\left(\alpha_1 c_2 c_3^2 + \alpha_2 c_3 + \alpha_3 c_2^2\right)} \end{aligned} \tag{11.151}$$

11.3.2 Center-Manifold Reduction

Again, we start by representing (11.128) as an operator differential equation; that is,

$$\dot{\boldsymbol{x}}_t = \mathcal{A}\boldsymbol{x}_t + \boldsymbol{F}(\boldsymbol{x}_t) \tag{11.152}$$

where $\boldsymbol{x}_t(\theta)\epsilon\mathcal{B}$ is defined by the shift operator

$$\boldsymbol{x}_t(\theta) = \boldsymbol{x}(t+\theta) \quad \text{for} \quad -\tau \le \theta \le 0$$

the linear operator $\mathcal{A}$ is defined by

$$\mathcal{A}\boldsymbol{p}(\theta) = \begin{cases} \dfrac{d}{d\theta}\boldsymbol{p}(\theta) & \text{for} \quad -\tau \le \theta \le 0 \\ L\boldsymbol{p}(0) - \beta_c R\boldsymbol{p}(-\tau) & \text{for} \quad \theta = 0 \end{cases} \tag{11.153}$$

and the operator $\boldsymbol{F}$ can be written as

$$\boldsymbol{F} = \begin{cases} 0 & \text{for} \quad -\tau \le \theta \le 0 \\ \boldsymbol{f} - \epsilon\delta R\boldsymbol{x}(t-\tau) & \text{for} \quad \theta = 0 \end{cases} \tag{11.154}$$

The adjoint operator associated with (11.153) is defined by

$$\mathcal{A}^*\boldsymbol{q}(\theta) = \begin{cases} -\dfrac{d}{d\theta}\boldsymbol{q}(\theta) & \text{for} \quad 0 \le \theta \le \tau \\ L^*\boldsymbol{q}(0) - \beta_c R^*\boldsymbol{q}(\tau) & \text{for} \quad \theta = 0 \end{cases} \tag{11.155}$$

and the "inner product" is defined in (11.24).

Equation 11.153 has an infinite number of eigenvalues λ_i and an infinite number of eigenfunctions $\boldsymbol{p}_i$. At $\beta = \beta_c$, the infinite-dimensional phase space of the solutions of $\dot{\boldsymbol{x}}_t = \mathcal{A}\boldsymbol{x}_t$ can be split into a two-dimensional center subspace, corresponding to the eigenvalues with zero real parts, and an infinite-dimensional stable subspace, corresponding to the eigenvalues with negative real parts. Consequently, the system (11.152) has a two-dimensional attractive subspace (the center manifold) and solutions of the system are locally attracted to this center manifold.

The center subspace of the linear operator $\mathcal{A}$ is spanned by the function satisfying the following boundary-value problem:

$$\frac{d}{d\theta}\boldsymbol{p}(\theta) = i\omega_c\boldsymbol{p}(\theta) \quad \text{for} \quad -\tau \le \theta \le 0 \tag{11.156}$$

$$L\boldsymbol{p}(0) - \beta_c R\boldsymbol{p}(-\tau) = i\omega_c\boldsymbol{p}(0) \tag{11.157}$$

The general solution of (11.156) can be written as

$$\boldsymbol{p}(\theta) = \boldsymbol{c}e^{i\omega_c\theta} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} e^{i\omega_c\theta} \tag{11.158}$$

where the c_i are constants. Substituting (11.158) into (11.157) yields

$$\left[L - i\omega_c I - \beta_c Re^{-i\omega_c\tau}\right]\boldsymbol{c} = 0 \tag{11.159}$$

Because $i\omega_c$ is an eigenvalue of (11.156) and (11.157), (11.159) has nontrivial solutions. Setting $c_1 = 1$, we find that c_2 and c_3 are given by (11.143).

Next, we calculate the adjoint at the critical value β_c. The eigenvalue corresponding to the adjoint is $-i\omega_c$. Therefore, it follows from (11.155) that

$$\frac{d}{d\theta}\boldsymbol{q}(\theta) = i\omega_c\boldsymbol{q}(\theta)$$

whose general solution can be expressed as

$$\boldsymbol{q}(\theta) = \boldsymbol{b}e^{i\omega_c\theta} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} e^{i\omega_c\theta} \tag{11.160}$$

Substituting (11.160) into (11.155) yields

$$\left[L^* + i\omega_c I - \beta_c R^* e^{i\omega_c\tau}\right] \boldsymbol{b} = 0 \tag{11.161}$$

whose determinant is zero because $i\omega_c$ is an eigenvalue of the original linear problem and hence the adjoint problem. Therefore, nontrivial solutions exist for the b_i; they are not unique. To uniquely determine the b_i, we require the "inner product" $\langle \boldsymbol{q}, \boldsymbol{p}\rangle = 1$. Consequently,

$$\bar{b}_1 + \bar{b}_2 c_2 + \bar{b}_3 c_3 - \beta_c \int_{-\tau}^{0} \left[\bar{b}_1 \bar{b}_2 \bar{b}_3\right] e^{-i\omega_c\tau} R \begin{Bmatrix} 1 \\ c_2 \\ c_3 \end{Bmatrix} d\xi = 1$$

or

$$\bar{b}_1 + \bar{b}_2 c_2 + \bar{b}_3 c_3 - \beta_c \tau \left(\alpha_1 \bar{b}_1 c_3 + \alpha_2 \bar{b}_2 c_2 + \alpha_3 \bar{b}_3\right) = 1 \tag{11.162}$$

Solving (11.161) and (11.162) yields

$$\bar{b}_1 = \frac{c_2 c_3}{3 c_2 c_3 - \beta_c \tau \left(\alpha_1 c_2 c_3^2 + \alpha_2 c_3 + \alpha_3 c_2^2\right)}, \quad \bar{b}_2 = \frac{\bar{b}_1}{c_2}, \quad \bar{b}_3 = \frac{\bar{b}_1}{c_3} \tag{11.163}$$

Having determined the center subspace and its adjoint, we decompose $\boldsymbol{x}_t(\theta)$ into two components: $y(t)\boldsymbol{p}(\theta) + \bar{y}(t)\bar{\boldsymbol{p}}(\dot{\theta})$ lying in the center subspace and the infinite-dimensional component $\boldsymbol{u}_t(\theta)$ transverse to the center subspace; that is,

$$\boldsymbol{x}_t(\theta) = y(t)\boldsymbol{p}(\theta) + \bar{y}(t)\bar{\boldsymbol{p}}(\theta) + \boldsymbol{u}_t(\theta) \tag{11.164}$$

where $\langle \boldsymbol{p}, \boldsymbol{u}\rangle = 0$ and $\langle \bar{\boldsymbol{p}}, \boldsymbol{u}\rangle = 0$. Substituting (11.164) into (11.152) yields

$$\begin{aligned} \dot{y}(t)\boldsymbol{p}(\theta) + \dot{\bar{y}}(t)\bar{\boldsymbol{p}}(\theta) + \dot{\boldsymbol{u}}_t(\theta) &= i\omega \boldsymbol{p}(\theta) y - i\omega \bar{\boldsymbol{p}}(\theta)\bar{y} + \mathcal{A}\boldsymbol{u}_t(\theta) \\ &\quad + \boldsymbol{F}\left(\boldsymbol{p} y + \bar{\boldsymbol{p}}\bar{y} + \boldsymbol{u}_t\right) \end{aligned} \tag{11.165}$$

Taking the "inner product" of (11.165) with $\boldsymbol{q}$, we obtain

$$\dot{y} = i\omega y + \langle \boldsymbol{q}, \boldsymbol{F}\rangle$$

Hence,

$$\begin{aligned} \dot{y} &= i\omega y - \epsilon\delta \left(\alpha_1 c_3 \bar{b}_1 + \alpha_2 \bar{b}_2 + \alpha_3 c_2 \bar{b}_3\right) e^{-i\omega_c\tau} y \\ &\quad - 3\left(\alpha_1 c_3^2 \bar{c}_3 \bar{b}_1 + \alpha_2 \bar{b}_2 + \alpha_3 c_2 \bar{c}_2 \bar{b}_3\right) y^2 \bar{y} y^2 \bar{y} + \text{HOT} + \text{NRT} \end{aligned} \tag{11.166}$$

Changing the notation and after some algebra, one can show that (11.166) is in full agreement with the result obtained by Liao, Guo, and Li (2007). Letting

$$y(t) = A(t) e^{i\omega_c t}$$

in (11.166) and using (11.163), we obtain (11.150) obtained by using the method of multiple scales.

11.4 Crane Control with Time-Delayed Feedback

In this section, we consider a model for controlling payload pendulation in a container crane using delayed position feedback. A polynomial approximation of the model can be expressed as (Nayfeh, 2006)

$$\begin{aligned}\ddot{\phi}(t) + \alpha_1\phi(t) + 2\mu\dot{\phi}(t) + k\ddot{\phi}(t-\tau) &= -\epsilon\alpha_3\phi^3(t) - \epsilon\alpha_4\phi(t)\dot{\phi}^2(t)\\ &- \epsilon\alpha_4\phi^2(t)\ddot{\phi}(t) + \epsilon k\phi(t-\tau)\dot{\phi}^2(t-\tau) - \epsilon k\alpha_5\phi^2(t)\ddot{\phi}(t-\tau)\\ &+ \tfrac{1}{2}\epsilon k\phi^2(t-\tau)\ddot{\phi}(t-\tau)\end{aligned} \tag{11.167}$$

where ϕ is the pendulation angle, k is the feedback gain, τ is the time delay, μ is the inherent damping coefficient, and the α_i are known constants. Taking k as the control parameter, one finds that the trivial solution of (11.167) undergoes successive Hopf bifurcations.

To determine the bifurcation values, we linearize (11.167) and obtain

$$\ddot{\phi}(t) + \alpha_1\phi(t) + 2\mu\dot{\phi}(t) + k\ddot{\phi}(t-\tau) = 0 \tag{11.168}$$

Next, we seek the solution of (11.168) in the form

$$\phi(t) = \phi_0 e^{(\sigma+i\omega)t} \tag{11.169}$$

where ϕ_0 is constant that depends on the initial conditions, and obtain

$$\alpha_1 + 2\mu(\sigma + i\omega) + (\sigma + i\omega)^2\left(1 + ke^{-(\sigma+i\omega)\tau}\right) = 0 \tag{11.170}$$

For a given k and τ, one can solve numerically (11.170) and obtain σ and ω. When $\sigma > 0$ the system response grows exponentially with time and the trivial solution is unstable; when $\sigma < 0$ the system response decays exponentially with time and the trivial solution is stable; and $\sigma = 0$ defines the stability boundary. To locate this boundary, we let $\sigma = 0$ in (11.170) and obtain

$$\alpha_1 + 2i\mu\omega_c - \omega_c^2(1 + k_c e^{-i\omega_c\tau}) = 0 \tag{11.171}$$

where ω_c and k_c locate the stability boundary for a given τ. Differentiating (11.170) with respect to $\sigma + i\omega$ and k, we obtain

$$\left.\frac{d(\sigma + j\omega)}{dk}\right|_{k=k_c} = -\frac{\omega_c^2}{k_c\omega_c^2\tau + 2ik_c\omega_c + 2(\mu + i\omega_c)e^{i\omega_c\tau}}$$

whose real part is different from zero. Therefore, the trivial solution loses stability with two complex conjugate eigenvalues transversely crossing the imaginary axis, and hence the trivial solution undergoes a Hopf bifurcation at $k = k_c$, and the stability boundary is the locus of Hopf bifurcations.

Using the method of multiple scales, we transform $\phi(t;\epsilon)$ and $\phi(t-\tau;\epsilon)$ as

$$\phi(t;\epsilon) \to \phi(T_0, T_1;\epsilon) \quad \text{and} \quad \phi(t-\tau;\epsilon) \to \phi(T_0 - \tau, T_1 - \epsilon\tau;\epsilon) \tag{11.172}$$

and seek a uniform first-order expansion of the solution of (11.167) in the form

$$\phi(t;\epsilon) = \phi_0(T_0, T_1) + \epsilon\phi_1(T_0, T_1) + \cdots \tag{11.173}$$

$$\phi(t-\tau;\epsilon) = \phi_0(T_0-\tau, T_1-\epsilon\tau) + \epsilon\phi_1(T_0-\tau, T_1-\epsilon\tau) + \cdots \tag{11.174}$$

Furthermore, we introduce a parameter δ to express the nearness of the gain k to the Hopf bifurcation value k_c, the critical gain, as

$$k = k_c + \epsilon\delta \tag{11.175}$$

Substituting (11.173)–(11.175) into (11.167) and equating coefficients of like powers of ϵ, we obtain

Order (ϵ^0)

$$D_0^2\phi_0 + k_c D_0^2\phi_{0\tau} + \alpha_1\phi_0 + 2\mu D_0\phi_0 = 0 \tag{11.176}$$

Order (ϵ)

$$\begin{aligned} D_0^2\phi_1 + k_c D_0^2\phi_{1\tau} + \alpha_1\phi_1 + 2\mu D_0\phi_1 &= -2D_0D_1\phi_0 - 2k_c D_0 D_1\phi_{0\tau} \\ &\quad - 2\mu D_1\phi_0 + \tau k_c D_0^2 D_1\phi_{0\tau} - \delta D_0^2\phi_{0\tau} - \alpha_3\phi_0^3 - \alpha_4\phi_0[D_0\phi_0]^2 \\ &\quad - \alpha_4\phi_0^2 D_0^2\phi_0 + k_c\phi_{0\tau}[D_0\phi_{0\tau}]^2 - k_c\alpha_5\phi_0^2 D_0^2\phi_{0\tau} + \tfrac{1}{2}k_c\phi_{0\tau}^2 D_0^2\phi_{0\tau} \end{aligned} \tag{11.177}$$

As in the preceding sections, the solution of (11.176) is taken to be

$$\phi_0 = Ae^{i\omega_c T_0} + \bar{A}e^{-i\omega_c T_0} \tag{11.178}$$

Substituting (11.178) into (11.177) yields

$$\begin{aligned} D_0^2\phi_1 + k_c D_0^2\phi_{1\tau} + \alpha_1\phi_1 + 2\mu D_0\phi_1 &= -bA'e^{i\omega_c T_0} + \delta\omega_c^2 Ae^{i\omega_c T_0 - i\omega_c\tau} \\ &\quad + \Lambda A^2\bar{A}e^{i\omega_c T_0} + \text{NST} + \text{cc} \end{aligned} \tag{11.179}$$

where

$$b = 2\mu + 2i\omega_c + 2i\omega_c k_c e^{-i\omega_c\tau} + \tau\omega_c^2 k_c e^{-i\omega_c\tau} \tag{11.180}$$

$$\Lambda = -3\alpha_3 + \tfrac{3}{2}k_c\omega_c^2 e^{-i\omega_c\tau} + 2\alpha_4\omega_c^2 + 2\alpha_5 k_c\omega_c^2 e^{-i\omega_c\tau} + \alpha_5 k_c\omega_c^2 e^{i\omega_c\tau} \tag{11.181}$$

Eliminating the terms that lead to secular terms from (11.179) leads to the normal form

$$A' = b^{-1}\delta\omega_c^2 e^{-i\omega_c\tau}A + b^{-1}\Lambda A^2\bar{A} \tag{11.182}$$

11.5 Exercises

11.5.1 Use the methods of multiple scales and normal forms to determine the normal form of

$$\ddot{u} + 2\mu\dot{u} + \omega^2 u + \alpha u^3 + k u(t - \tau) = 0$$

11.5.2 Use the methods of multiple scales and normal forms to determine the normal form of

$$\ddot{u} + \omega^2 u = \mu(\dot{u} - u^2\dot{u}) + k u(t - \tau)$$

where $\mu > 0$. Determine the range of gain k and time delay τ for which the limit cycles are quenched.

11.5.3 Consider the system

$$\ddot{u} + \omega^2 u + 2\mu\dot{u} + \alpha u^3 + k u(t - \tau) = f \cos \Omega t$$

Determine the resonance frequency for the linear system. Then, determine an approximation to the nonlinear system response for values of Ω near the resonance frequency.

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